Measuring the Value of Randomized Solutions in Distributionally Robust Optimization

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CRC in Decision making under uncertainty

joint work with Ahmed Saif (Dalhousie U.)

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Summary of Wolfram's Talk
Axiomatic Motivation for DRO

Definition: A risk measure ρ_0 is called ambiguity averse if it satisfies for all $X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$:

- ☑ **Ambiguity aversion:** If $\{F_X^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\} \subseteq \{F_Y^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\}$, then $\rho_0(X) \leq \rho_0(Y)$.
- ☑ **Ambiguity monotonicity:** If $\varrho_0(F_X^{\mathbb{P}}) \leq \varrho_0(F_Y^{\mathbb{P}})$ for all $\mathbb{P} \in \mathcal{P}_0$, then $\rho_0(X) \leq \rho_0(Y)$.

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is ambiguity averse and translation invariant.

Then the risk measure satisfies

$$\rho_0(X) = \sup_{\mathbb{P} \in \mathcal{P}_0} \varrho_0(F_X^{\mathbb{P}}) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0).$$

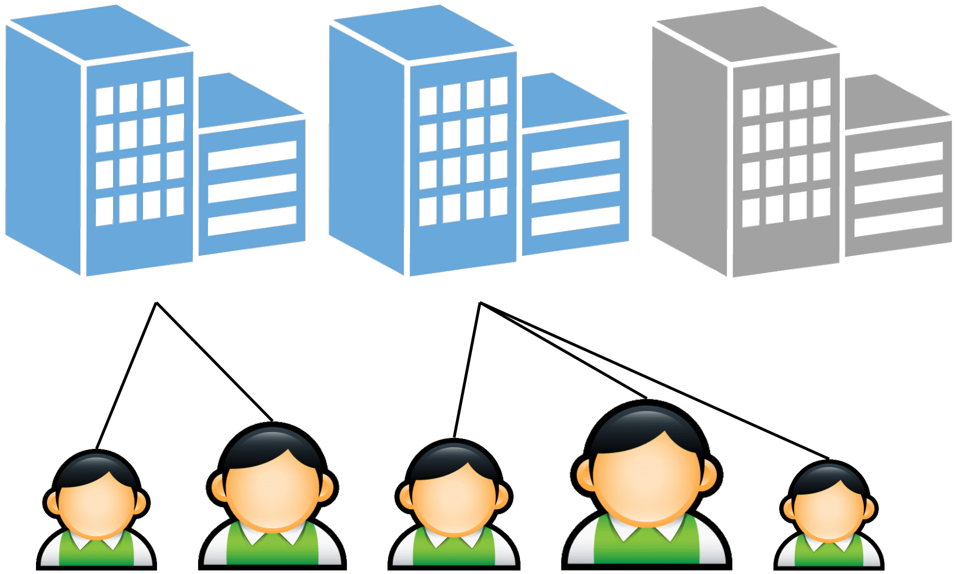
Summary of Wolfram's Talk

Randomisation proof/receptiveness

	Randomization Receptive	Randomization Proof
Distributional Ambiguity	every RM with Lebesgue property	convex RM's with convex $\mathcal{X}_0$
Stochastic Uncertainty	M… me… / M… mean semi-deviati…	convex RM's … $\mathcal{X}_0$ / …asi- …RM's

In DRO problems, if the set of reachable random cost is non-convex, then a <u>randomized strategy</u> might outperform <u>all deterministic ones</u>.

Distributionally Robust Uncapacitated Facility Location Problem



$\mathcal{J}$: set of potential facility locations
$\mathcal{I}$: set of customers

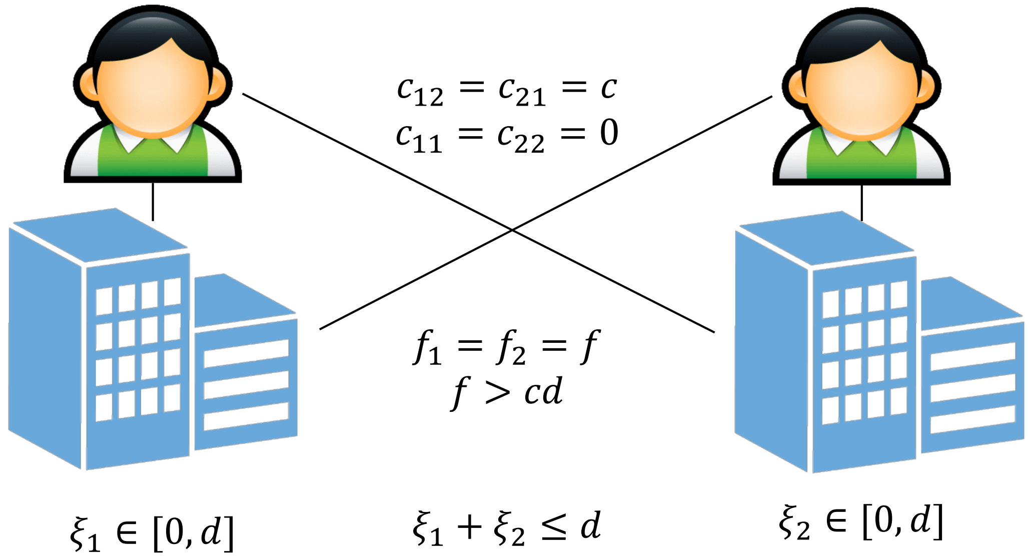
f_j: setup cost of facility j
ξ_i: demand of customer i
c_{ij}: unit shipping cost from j to i

$x_j = 1$ if facility j is opened, 0 otherwise
$y_{ij} = 1$ if customer i's demand is served by facility j

The UFLP is formulated as:

$$\begin{aligned}
\underset{x,y}{\text{minimize}} \quad & \max_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi} \left[\sum_{j \in \mathcal{J}} f_j x_j + \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \xi_i c_{ij} y_{ij} \right] & \\
\text{subject to} \quad & \sum_{j \in \mathcal{J}} y_{ij} = 1 \qquad , \ \forall i \in \mathcal{I} & (1) \\
& y_{ij} \le x_j \qquad , \ \forall i \in \mathcal{I}, \, j \in \mathcal{J} & (2) \\
& x \in \{0, 1\}^{|\mathcal{J}|} \, , \; y \in \{0, 1\}^{|\mathcal{I}| \times |\mathcal{J}|} & (3)
\end{aligned}$$

Illustrative Example



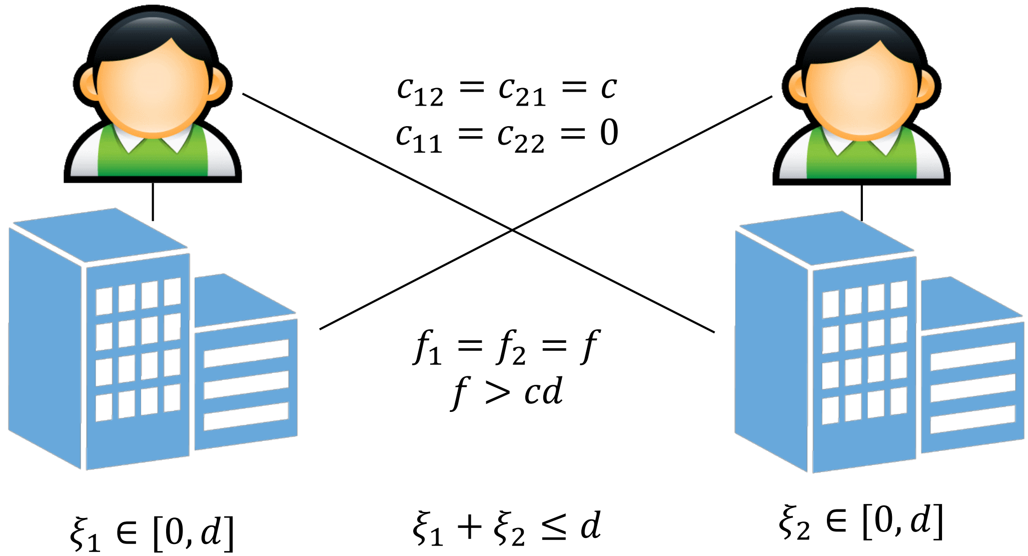
Without randomization

- $v_d^* = \min_{x,y} \sup_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi}[f(x_1 + x_2) + c(\xi_1 y_{12} + \xi_2 y_{21})]$
- Solution 1:
  $x_1^{*1} = 1,\ y_{21}^{*1} = 1,\ v^{*1} = f + cd$
- Solution 2:
  $x_2^{*2} = 1,\ y_{12}^{*2} = 1,\ v^{*2} = f + cd$
- Optimal value : $v_d^* = f + cd$, the worst-case distribution puts all the mass on the wrong node.

With randomization

- $v_r^* = \min_{p_1, p_2} \sup_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi}[\sum_k p_k(f(x_1^{*k} + x_2^{*k}) + c(\xi_1 y_{12}^{*k} + \xi_2 y_{21}^{*k}))]$
- Randomized solution
  $p^* : p_1^* = p_2^* = 0.5$
- Randomized optimal value :
  $v_r^* = f + cd/2$
- Randomization ensures that each unit of demand has 50% chance of staying at same node.

Illustrative Example



Without randomization

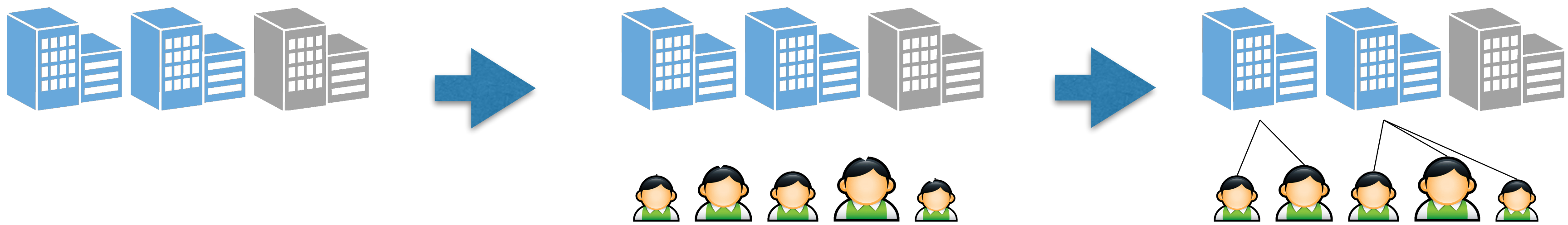
- $v_d^* = \min_{x,y} \sup_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi}[f(x_1 + x_2) + c(\xi_1 y_{12} + \xi_2 y_{21})]$
- Solution 1: $x_1^{*1} = 1,\ y_{21}$ ...
- Solution 2: $x_2^{*2} = 1,\ y_{12}$ ...
- Optimal value : ... worst-case distribution p... the mass on the wrong node.

With randomization

- $v_r^* = \min_{p_1, p_2} \sup_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi}[\sum_k p_k(f(x_1^{*k} + x_2^{*k}) + c(\xi_1 y_{12}^{*k} + \xi_2 y_{21}^{*k}))]$
- ...dom... solution
- ...al value :
- ...nsures that each unit ... mand has 50% chance of staying at same node.

Randomization reduces worst-case expected cost by «cd/2» (i.e. up to 25%)

Two-Stage Distributionally Robust Capacitated Facility Location Problem



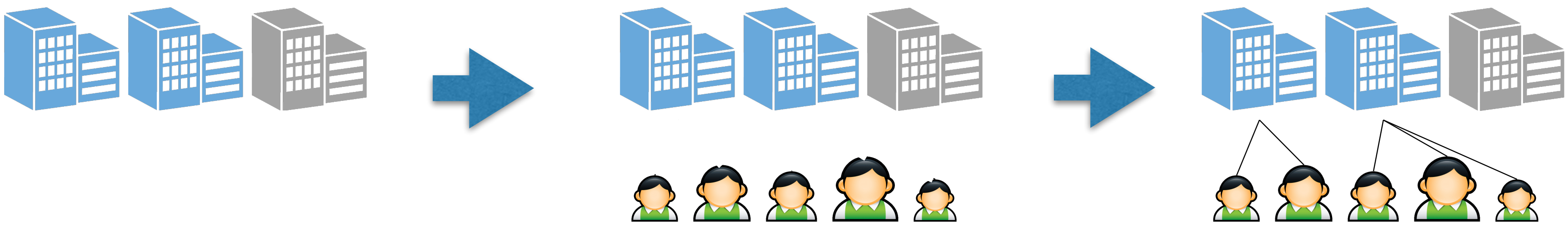
The Two-Stage DR CFLP is formulated as:

$$\underset{x \in \{0,1\}^{|\mathcal{J}|}}{\text{minimize}} \quad \max_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi} \left[\sum_{j \in \mathcal{J}} f_j x_j + G(x, \xi) \right]$$

where

$$\begin{aligned} G(x,\xi) \; := \quad \min_{z \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}} \quad & \sum_{i,j} c_{i,j} z_{i,j} \\ \text{subject to} \quad & \sum_{j} z_{i,j} \geq \xi_i \qquad \forall\, i \in \mathcal{I} \\ & \sum_{i} z_{i,j} \leq V_j x_j \qquad \forall\, j \in \mathcal{J} \end{aligned}$$

Two-Stage Distributionally Robust Capacitated Facility Location Problem



The Two-Stage DR CFLP is formulated as:

$$\underset{x\in\{0,1\}^{|\mathcal{J}|}}{\text{minimize}} \quad \max_{F_\xi\in\mathcal{D}} \mathbb{E}_{F_\xi}\left[\sum_{j\in\mathcal{J}} f_j x_j + G(x,\xi)\right]$$

where

$$G(x,\xi) \;:=\; \min_{z\in\mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|}} \quad \sum_{i,j} c_{i,j} z_{i,j}$$

How much improvement can be achieved using randomization here ???

$$\forall i \in \mathcal{I}$$

$$\sum_i z_{i,j} \le V_j x_j \qquad \forall j \in \mathcal{J}$$

The Value of Randomized Solutions

We define the pure strategy problem

$$v_d := \min_{x \in \mathcal{X}_0} \sup_{F_\xi \in \mathcal{D}} \rho_{\xi \sim F_\xi} [h(x, \xi)] \qquad \text{(PSP)}$$

We define the randomized strategy problem

$$v_r := \min_{F_x \in \Delta(\mathcal{X}_0)} \sup_{F_\xi \in \mathcal{D}} \rho_{(X,\xi) \sim F_x \times F_\xi} [h(X, \xi)] \qquad \text{(RSP)}$$

We define the value of randomized solutions

$$\text{VRS} := v_d - v_r$$

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   - ☑ Bound obtained from the Convex Relaxations
   - ☑ Implications for Totally Unimodular MILP
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4. Numerical experiments
   - ☑ DR Assignment Problems
   - ☑ DR Uncapacitated Facility Location Problems
   - ☑ Two-Stage DR Capacitated Facility Location Problems

Bounding VRS using Convex Hull of $\mathcal{X}_0$

Theorem 1. *Given that* ρ *is a convex risk measure and* $h(x,\xi)$ *a convex function with respect to* x *for all* $\xi \in \mathbb{R}^m$. *Let* $\mathcal{X}$ *be any set known to contain the convex hull of* $\mathcal{X}_0$, *then*

$$VRS \leq v_d - \min_{x\in\mathcal{X}} \sup_{F_\xi \in \mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi)) \, .$$

Bounding VRS using Convex Hull of $\mathcal{X}_0$

Theorem 1. *Given that ρ is a convex risk measure and $h(x,\xi)$ a convex function with respect to x for all $\xi \in \mathbb{R}^m$. Let $\mathcal{X}$ be any set known to contain the convex hull of $\mathcal{X}_0$, then*

$$VRS \leq v_d - \min_{x\in\mathcal{X}} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi))\,.$$

Proof:

$$\begin{aligned}
\min_{x\in\mathcal{X}} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi)) &\leq \min_{x\in\mathcal{C}(\mathcal{X}_0)} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi)) \\
&= \min_{g(\cdot)\in\mathcal{G}} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(g(\xi)) \\
&= \min_{F_g\in\Delta(\mathcal{G})} \sup_{F_\xi\in\mathcal{D}} \rho_{(G,\xi)\sim F_g\times F_\xi}(G(\xi)) \\
&= \min_{F_g\in\Delta(\mathcal{G}_0)} \sup_{F_\xi\in\mathcal{D}} \rho_{(G,\xi)\sim F_g\times F_\xi}(G(\xi)) \\
&= \inf_{F_x\in\Delta(\mathcal{C}(\mathcal{X}_0))} \sup_{F_\xi\in\mathcal{D}} \rho_{(X,\xi)\sim F_x\times F_\xi}(h(X,\xi)) \\
&\leq \inf_{F_x\in\Delta(\mathcal{X}_0)} \sup_{F_\xi\in\mathcal{D}} \rho_{(X,\xi)\sim F_x\times F_\xi}[h(X,\xi)] = v_r\,,
\end{aligned}$$

where

$$\mathcal{G} := \{g : \mathbb{R}^m \to \mathbb{R} \,|\, \exists x \in \mathcal{C}(\mathcal{X}_0),\ g(\xi) \geq h(x,\xi) \forall\, \xi\}$$

$$\mathcal{G}_0 := \{g : \mathbb{R}^m \to \mathbb{R} \,|\, \exists x \in \mathcal{C}(\mathcal{X}_0),\ g(\xi) = h(x,\xi) \forall\, \xi\}$$

Bounding VRS using Convex Hull of $\mathcal{X}_0$

Theorem 1. *Given that ρ is a convex risk measure and $h(x,\xi)$ a convex function with respect to x for all $\xi \in \mathbb{R}^m$. Let $\mathcal{X}$ be any set known to contain the convex hull of $\mathcal{X}_0$, then*

$$VRS \leq v_d - \min_{x\in\mathcal{X}} \sup_{F_\xi \in \mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi))\,.$$

Proof:

$$\min_{x\in\mathcal{X}} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi)) \leq \min_{x\in\mathcal{C}(\mathcal{X}_0)} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(h(x,\xi))$$

$$= \min_{g(\cdot)\in\mathcal{G}} \sup_{F_\xi\in\mathcal{D}} \rho_{\xi\sim F_\xi}(g(\xi))$$

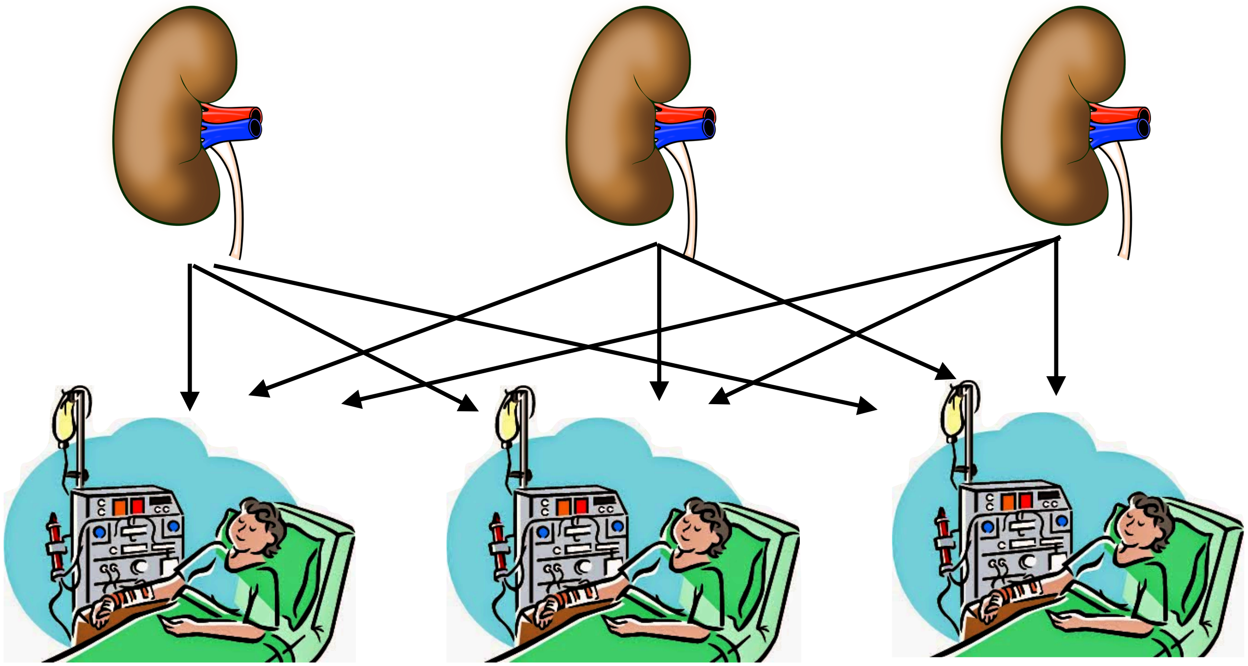
$$= \min_{F_g\in\Delta(\mathcal{G})} \sup_{F_\xi\in\mathcal{D}} \rho_{(G,\xi)\sim F_g\times F_\xi}(G(\xi))$$

$$= \min_{F_g\in\Delta(\mathcal{G}_0)} \sup_{F_\xi\in\mathcal{D}} \rho_{(G,\xi)\sim F_g\times F_\xi}(G(\xi))$$

$$= \inf \; \sup \; \rho_{(X,\xi)\sim F_x\times F_\xi}(h(X,\xi))$$

If $\mathcal{X} = \mathcal{C}(\mathcal{X}_0)$, the risk measure is the expected value, and the cost function is linear in x, then this bound is tight and is achieved by any $F_x^* \in \Delta(\mathcal{X}_0) : E_{X\sim F_x^*}[X] = x^*$.

Distributionally Robust Assignment Problem



$\mathcal{J}$: set of potential recipient

$\mathcal{I}$: set of donated organs

$\xi_{\mathbf{i,j}}$: **incompatibility score between organ i and recipient j**

$\mathbf{x_{i,j}} = \mathbf{1}$ **if we assign organ i to recipient j**

The DR Assignment Problem is formulated as:

$$\min_{x \in \mathcal{X}_{\text{AP}}} \max_{F_\xi \in \mathcal{D}} \mathbb{E}_{\xi \sim F_\xi} \left[\sum_{i \in I} \sum_{j \in J} \xi_{ij} x_{ij} \right]$$

where

$$\mathcal{X}_{\text{AP}} := \left\{ x \in \{0,\, 1\}^{n_i \times n_j} \;\middle|\; \begin{array}{l} \sum_{j \in J} x_{ij} = 1\,,\ \forall\, i \\ \sum_{i \in I} x_{ij} = 1\,,\ \forall\, j \end{array} \right\}$$

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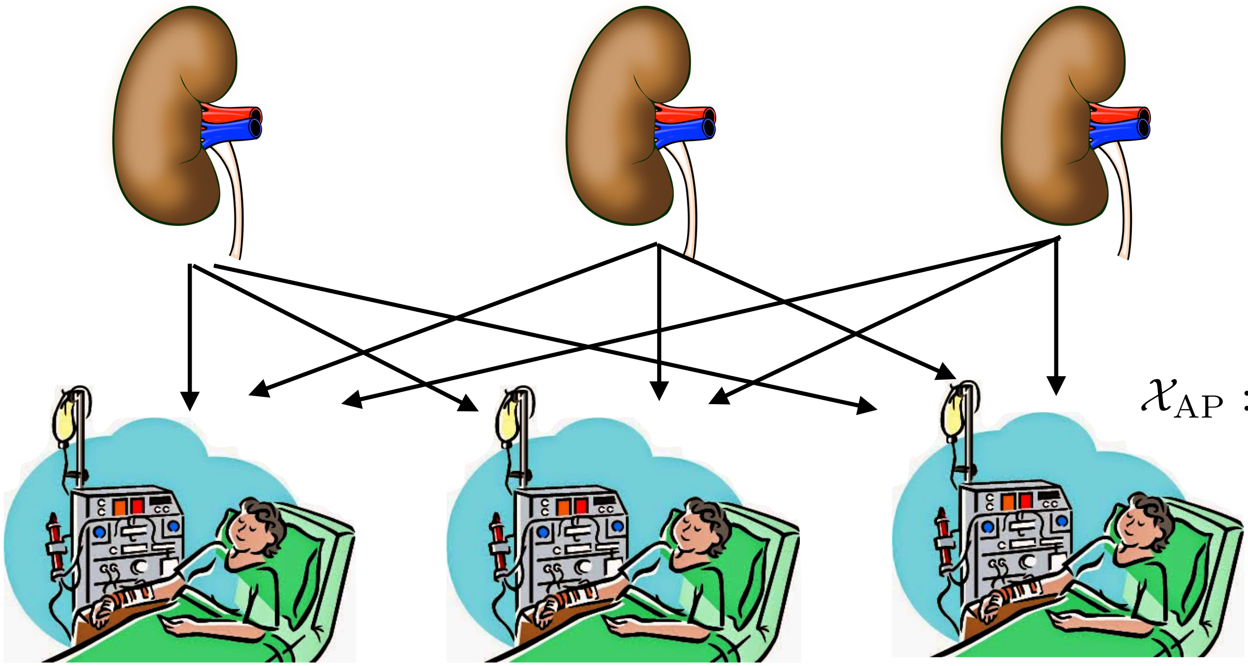
VRS for DR Assignment Problem



$$\min_{x \in \mathcal{X}_{\text{AP}}} \max_{F_\xi \in \mathcal{D}} \mathbb{E}_{\xi \sim F_\xi} \left[\sum_{i \in I} \sum_{j \in J} \xi_{ij} x_{ij} \right]$$

$$\mathcal{X}_{\text{AP}} := \left\{ x \in \{0,\, 1\}^{n_i \times n_j} \;\middle|\; \begin{array}{l} \sum_{j \in J} x_{ij} = 1\,,\ \forall\, i \\ \sum_{i \in I} x_{ij} = 1\,,\ \forall\, j \end{array} \right\}$$

$$\text{VRS} = \min_{x \in \mathcal{X}_{\text{AP}}} \delta^*(x|\mathcal{U}) - \min_{x' \in \mathcal{X}'_{\text{AP}}} \delta^*(x'|\mathcal{U})$$

where

$$\left\{ \qquad \middle| \; \sum x_{ij} = 1\,,\ \forall\, i \right\}$$

An optimal randomized strategy can be obtained by solving: $\underset{p \in \Delta}{\text{minimize}} \, \|x'^* - \sum_{k \in \mathcal{K}} p_k x^k\|$.

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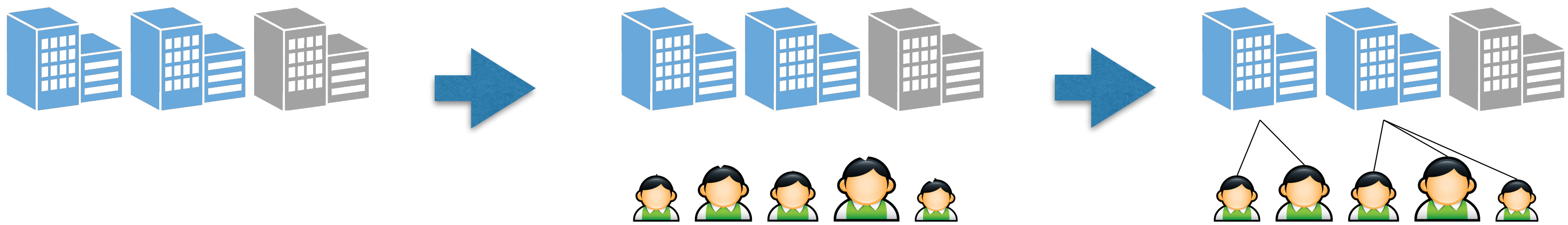
   - ☑ Two-Stage DR Capacitated Facility Location Problems

Two-Stage Distributionally Robust Capacitated Facility Location Problem



The Two-Stage DR CFLP is formulated as:

$$\underset{x \in \{0,1\}^{|\mathcal{J}|}}{\text{minimize}} \quad \max_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi} \left[\sum_{j \in \mathcal{J}} f_j x_j + G(x, \xi) \right]$$

where

$$\begin{aligned} G(x,\xi) \;:=\; & \min_{z \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}} && \sum_{i,j} c_{i,j} z_{i,j} \\ & \text{subject to} && \sum_{j} z_{i,j} \geq \xi_i && \forall\, i \in \mathcal{I} \\ & && \sum_{i} z_{i,j} \leq V_j x_j && \forall\, j \in \mathcal{J} \end{aligned}$$

Two-Stage Distributionally Robust Linear Program with RHS Uncertainty

A Two-Stage DR LP with RHS uncertainty is formulated as:

$$\underset{x\in\mathcal{X}\subseteq\{0,1\}^n}{\text{minimize}} \ \sup_{F_\xi\in\mathcal{D}} \ \mathbb{E}_{\xi\sim F_\xi}[c_1^T x + G(x,\xi)]$$

where

$$G(x,\xi) := \min_{y\in\mathbb{R}^m} \ c_2^T y$$

$$\text{s.t.} \ \ Ax + By \leq W\xi + b$$

We wish to identify the optimal randomized strategy

$$\underset{F_x\in\Delta(\mathcal{X})}{\text{minimize}} \ \sup_{F_\xi\in\mathcal{D}} \ \mathbb{E}_{(X,\xi)\sim F_x\times F_\xi}[c_1^T X + G(X,\xi)]$$

which can be simplified since feasible set $\mathcal{X}$ is discrete

$$\underset{p\in\Delta\subset\mathbb{R}_+^{|\mathcal{X}|}}{\text{minimize}} \ \sup_{F_\xi\in\mathcal{D}} \ \sum_{k\in\mathcal{K}} p_k c_1^T x^k + \mathbb{E}_{\xi\sim F_\xi}[\underbrace{\sum_{k\in\mathcal{K}} p_k G(x^k,\xi)}_{g(p,\xi)}]$$

Under the Wasserstein Ambiguity Set

Given a sample set $\{\hat{\xi}_1, \ldots, \hat{\xi}_N\}$, we assume the ambiguity set is a Wasserstein ball:

$$\mathcal{D}(\epsilon) := \left\{ F_\xi \;\middle|\; \begin{array}{l} \mathbb{P}_{F_\xi}(\xi \in \Xi) = 1 \\ d_W(F_\xi, \hat{F}_\xi) \leq \epsilon \end{array} \right\}$$

Based on Mohajerin Esfahani & Kuhn (2017), the DRO can be reformulated as:

$$\begin{array}{ll} \underset{p\in\Delta,\lambda\in\mathbb{R},s}{\text{minimize}} & \displaystyle\sum_{k\in\mathcal{K}} c_1^T x^k p_k + \lambda\epsilon + (1/N)\sum_n s_n \\ \text{subject to} & \displaystyle\sup_{\xi\in\Xi} g(p,\xi) - \lambda\|\xi - \xi_n\| \leq s_n \;,\; \forall n \\ & \displaystyle\sum_{k\in\mathcal{K}} p_k = 1 \end{array}$$

Under the Wasserstein Ambiguity Set

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Based on Mohajerin Esfahani & Kuhn (2017), the DRO can be reformulated as:

Equivalently as:

$$\begin{aligned} \underset{p \in \Delta, \lambda \in \mathbb{R}, s}{\text{minimize}} \quad & \sum_{k \in \mathcal{K}} c_1^T x^k p_k + \lambda \epsilon + (1/N) \sum_n s_n \\ \text{subject to} \quad & \sup_{(\xi, \gamma) \in \Xi'_n} g(p, \xi) - \lambda \gamma \leq s_n \,, \; \forall n \\ & \sum_{k \in \mathcal{K}} p_k = 1 \end{aligned}$$

where

$$\Xi'_n := \{(\xi, \gamma) | \xi \in \Xi, \; \|\xi - \hat{\xi}_n\| \leq \gamma\}$$

Formulating a Large-scale LP

When 1-norm is used and Ξ is polyhedral, each Ξ'_n is polyhedral, since $g(p,\cdot)$ is convex, one can instead enumerate the vertices

$$\begin{aligned}
\underset{p\in\Delta,\lambda\in\mathbb{R},s}{\text{minimize}} \quad & \sum_{k\in\mathcal{K}} c_1^T x^k p_k + \lambda\epsilon + (1/N)\sum_n s_n \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}} p_k G(x^k, \xi^{h_n}) - \lambda\gamma^{h_n} \leq s_n \ , \ \forall n \ , \ \forall\, h_n \in \mathcal{H}_n \\
& \sum_{k\in\mathcal{K}} p_k = 1
\end{aligned}$$

One can solve this large scale linear program using a two-layer column and constraint generation

Formulating a Large-scale LP

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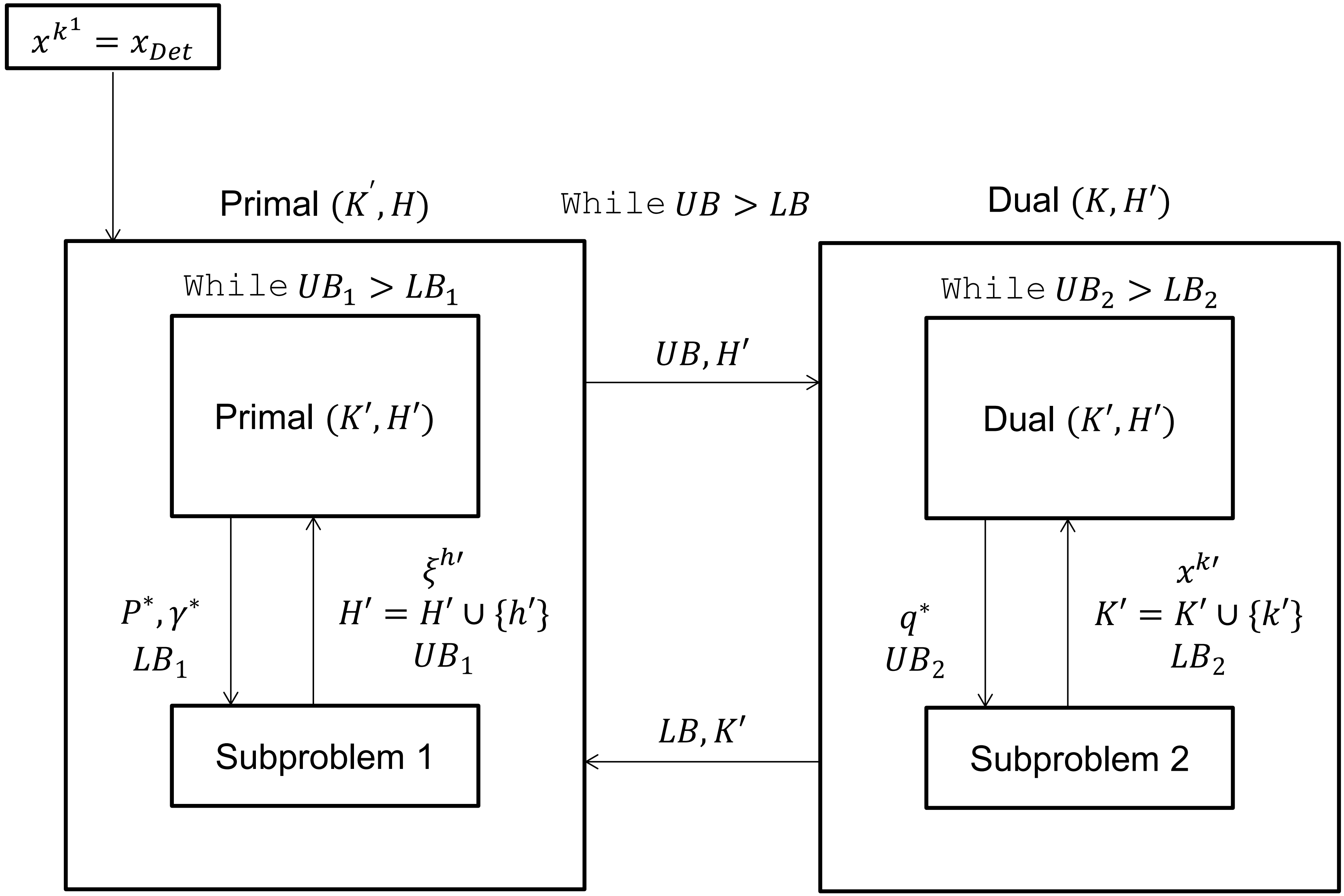
\underset{p\in\Delta,\lambda\in\mathbb{R},s}{\text{minimize}} \quad & \sum_{k\in\mathcal{K}'} c_1^T x^k p_k + \lambda\epsilon + (1/N)\sum_n s_n \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}'} p_k G(x^k, \xi^{h_n}) - \lambda\gamma^{h_n} \leq s_n \;,\; \forall n \;,\; \forall\, h_n \in \mathcal{H}'_n \\
& \sum_{k\in\mathcal{K}'} p_k = 1
\end{aligned}$$

One can solve this large scale linear program using a two-layer column and constraint generation

$$\mathcal{K}' \rightarrow \mathcal{K} \qquad \mathcal{H}' := \{\mathcal{H}'_n\}_{n=1}^N \rightarrow \mathcal{H} := \{\mathcal{H}_{n=1}^N\}_n \;,\; \forall\, n$$

Two-layer Column Generation Algorithm



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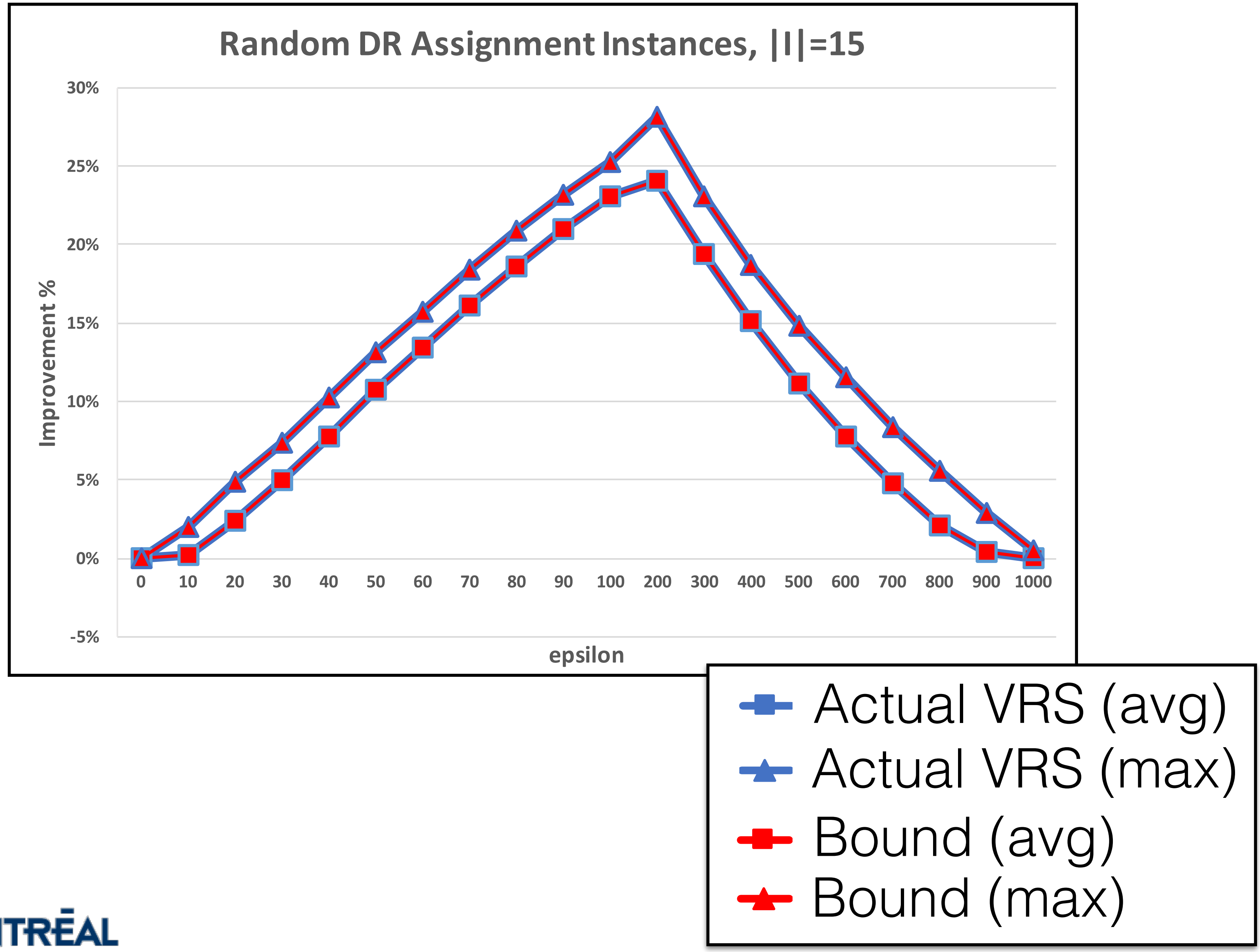
Quality of Bound & True VRS



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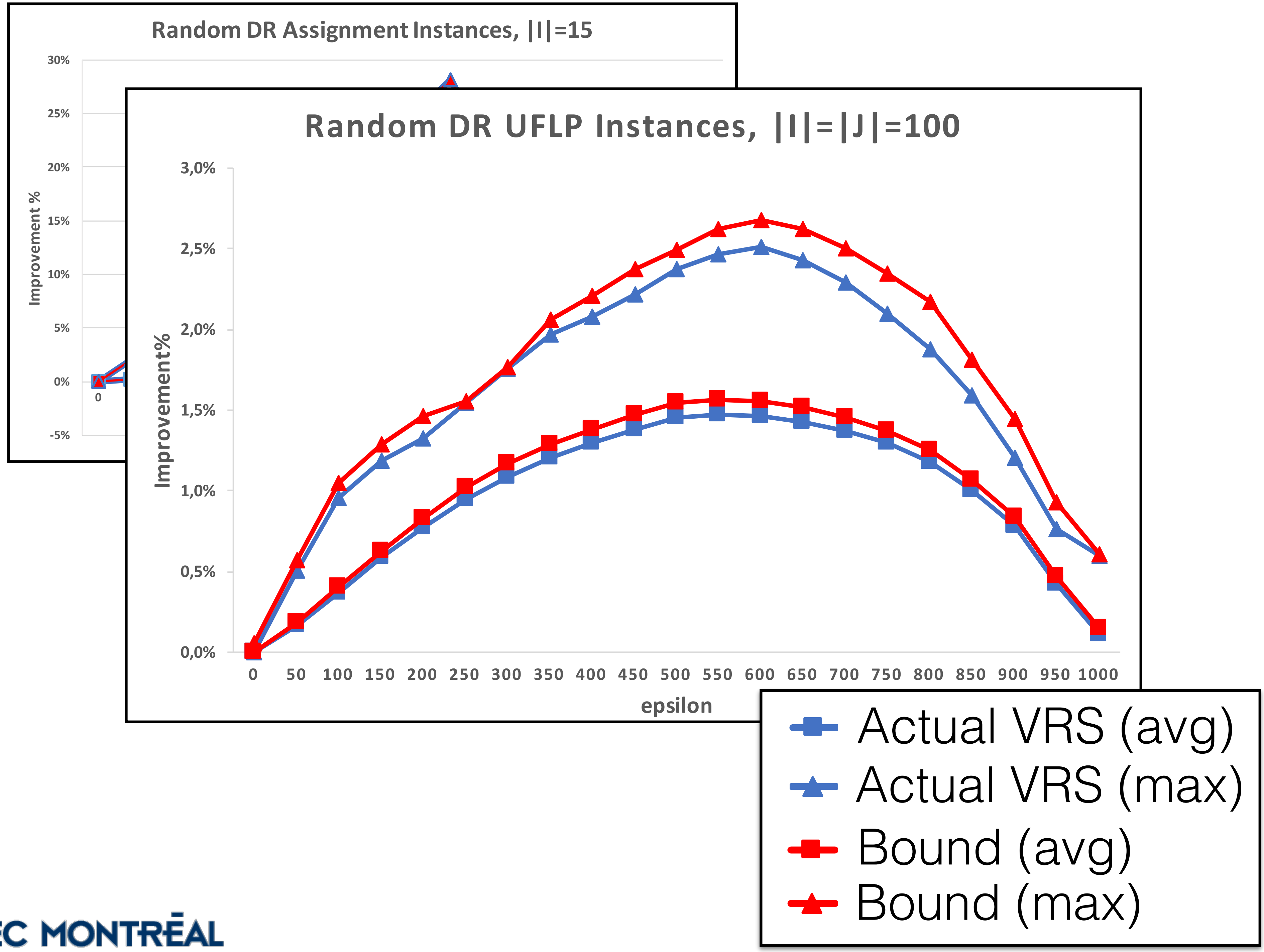

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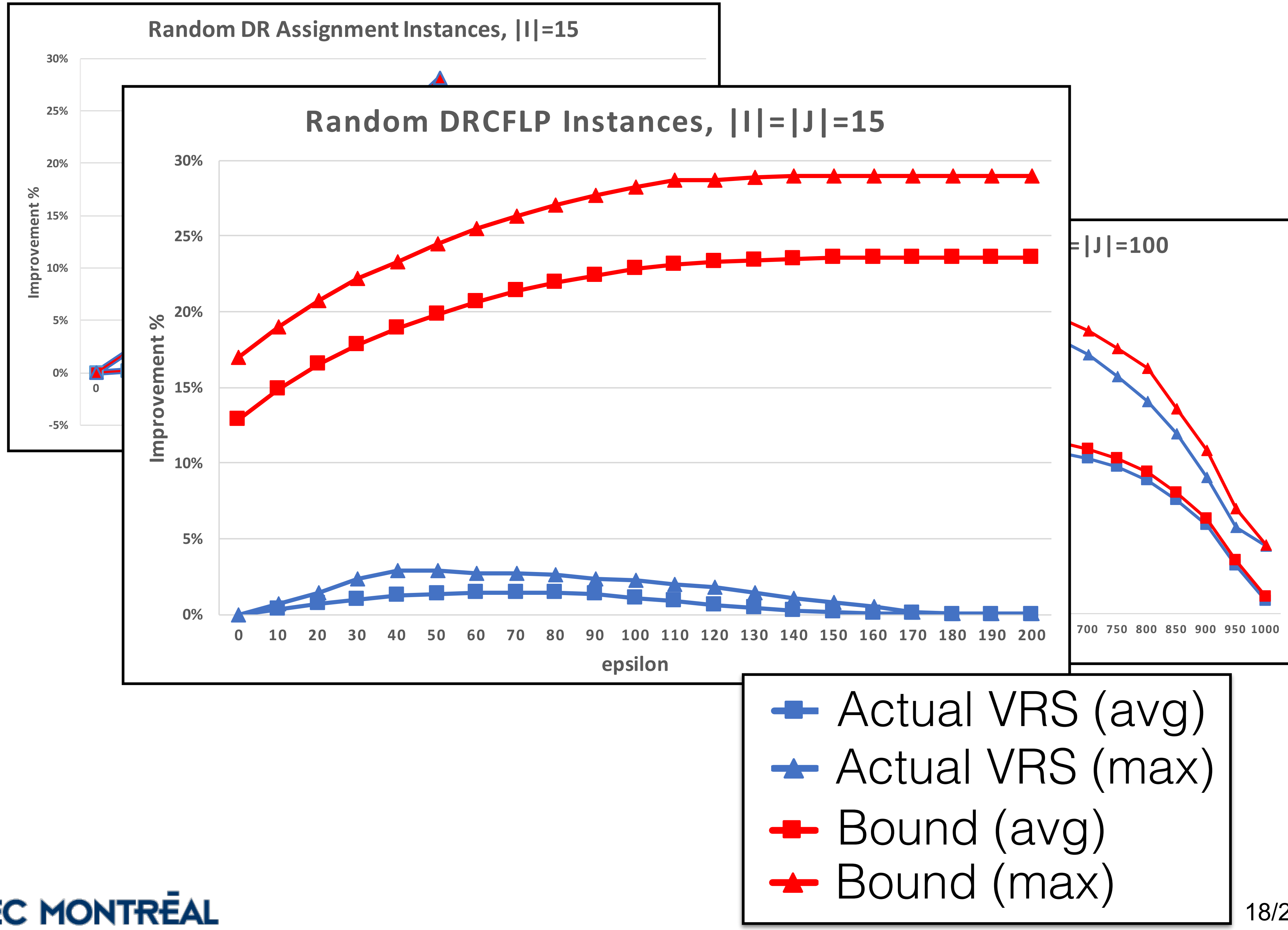
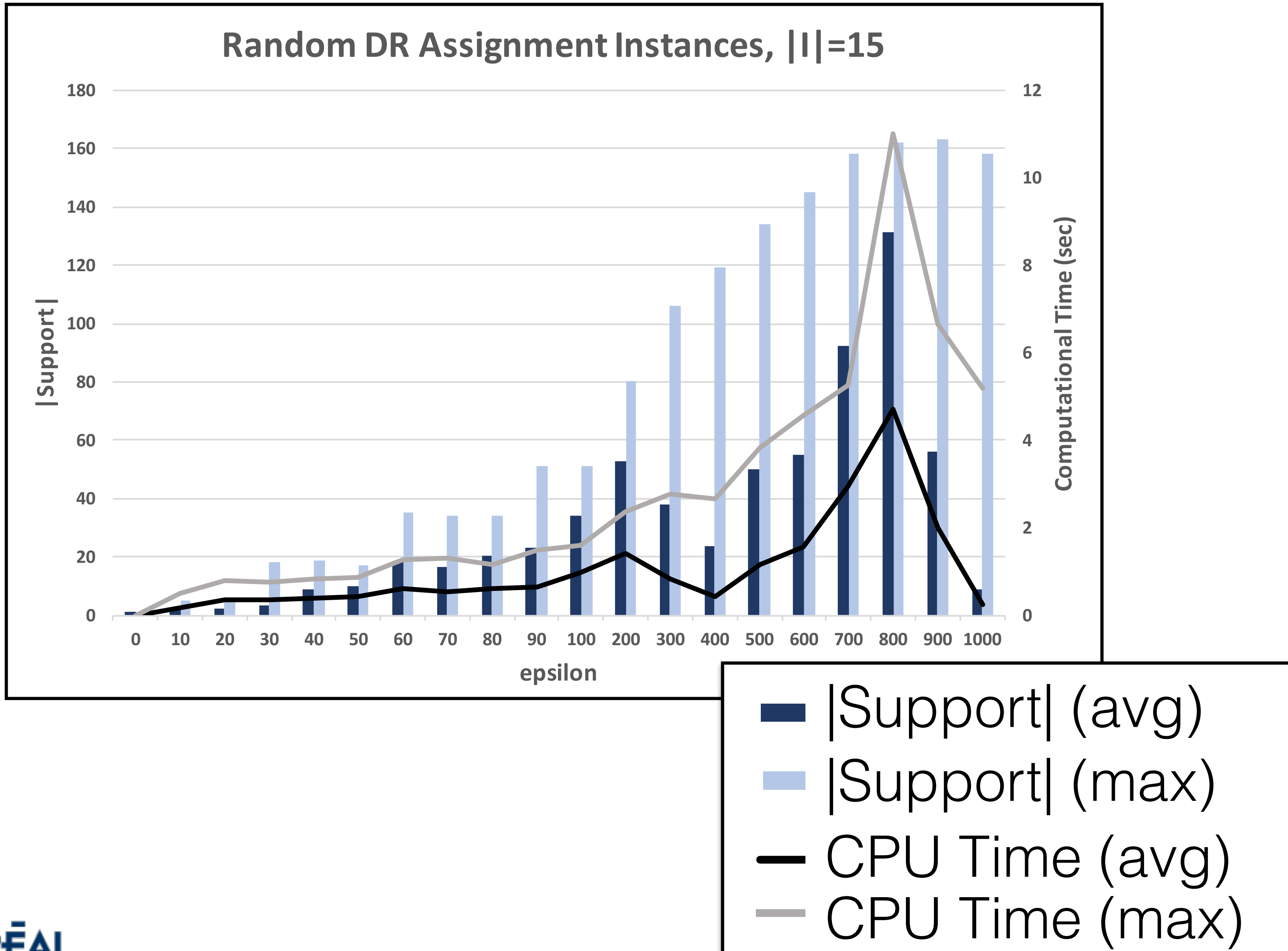
Quality of Bound & True VRS



Optimal Support Size & Computation Time



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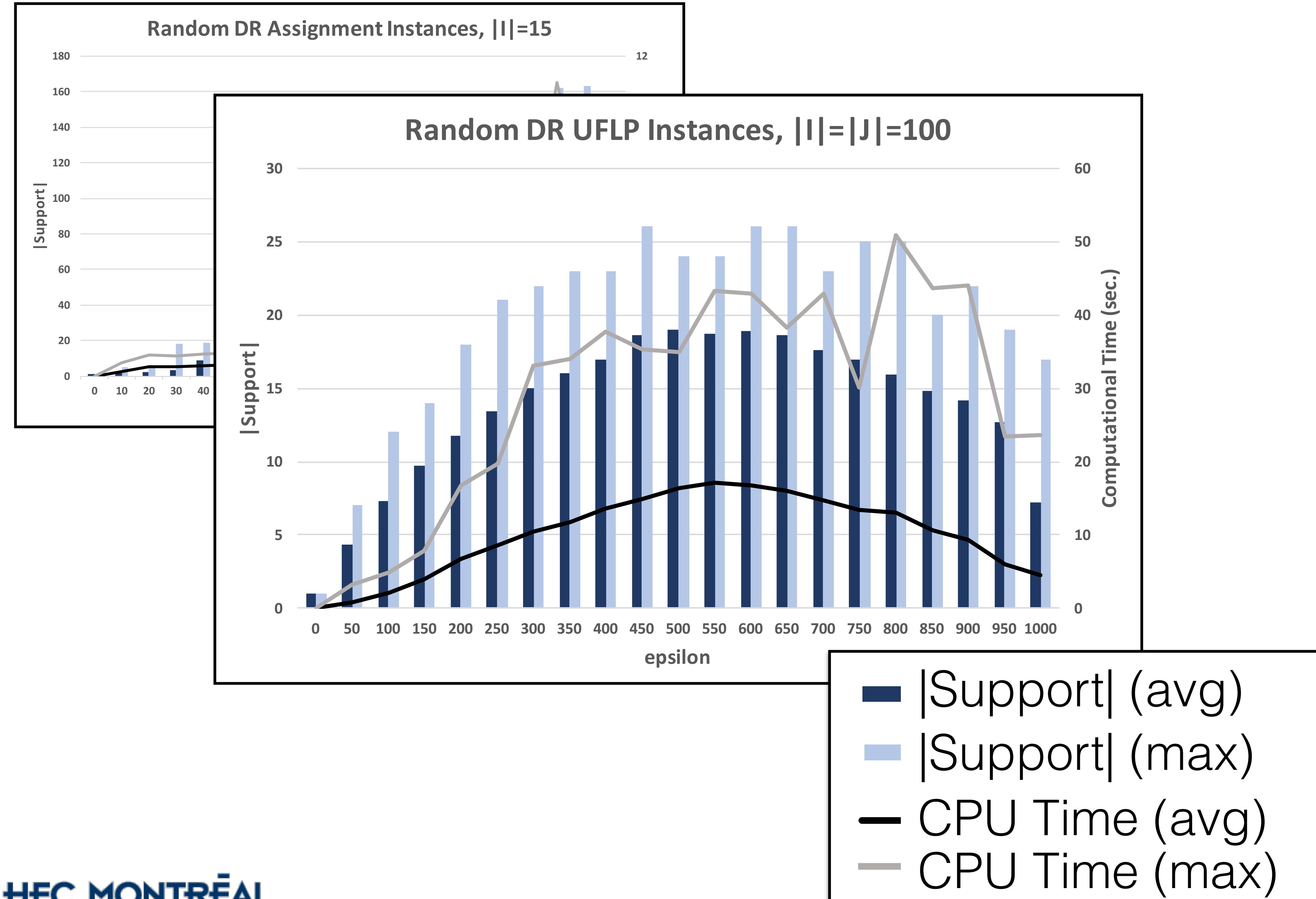

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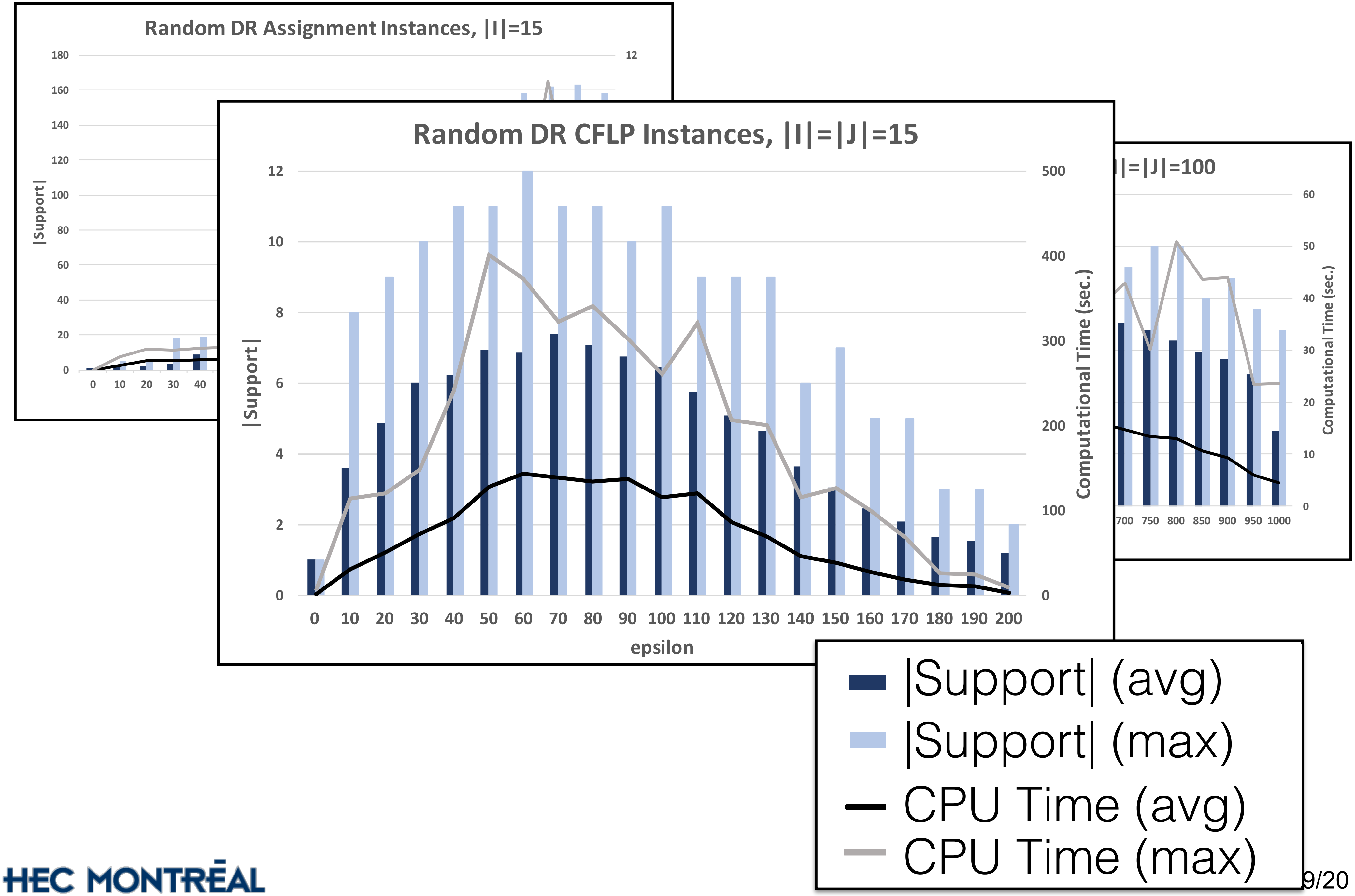
Optimal Support Size & Computation Time



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Randomized decisions in SP and DRO:

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