# Linearized Robust Counterparts of Two-Stage Distribution Problems

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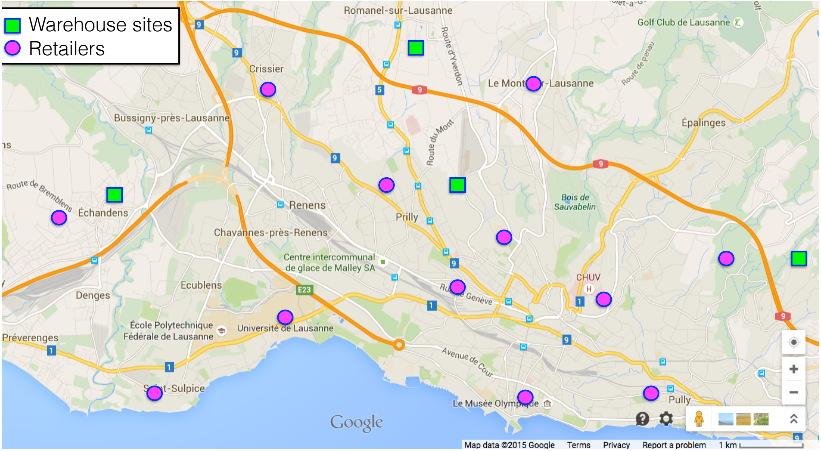
# A Classical Distribution Problem

- Facility location-transportation model



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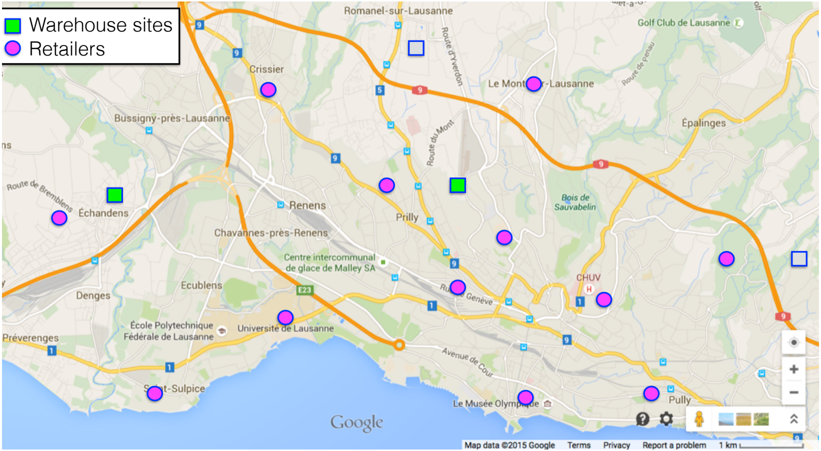

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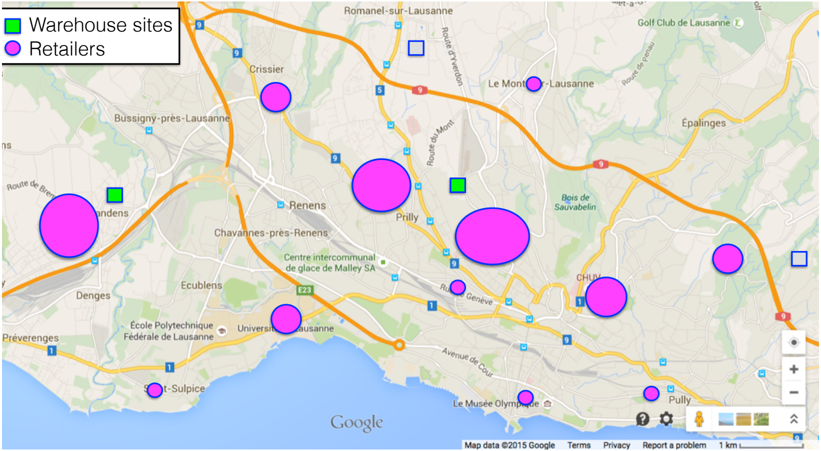

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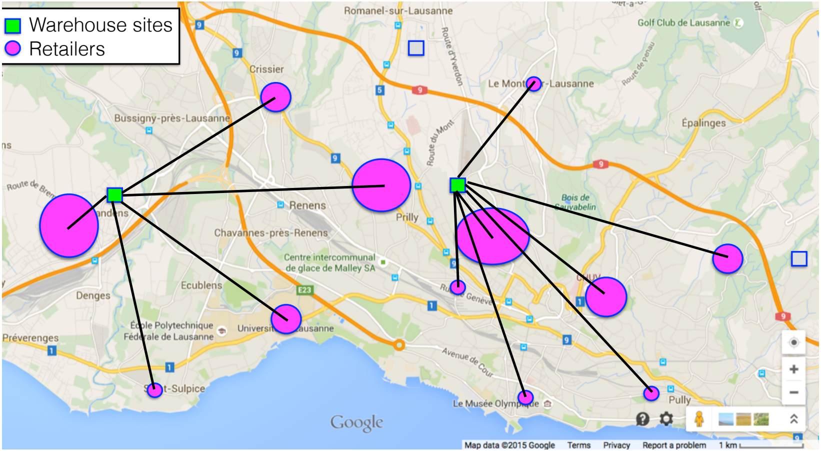

# A Classical Distribution Problem

- Facility location-transportation model

$$\max_{I\in\{0,1\}^n, x, Y\geq 0} \quad \eta \overbrace{\sum_i \sum_j Y_{ij}}^{\text{sold product}} - \left( \overbrace{c^T x + K^T I}^{\text{location cost}} + \overbrace{\sum_i \sum_j (p_i + t_{ij}) Y_{ij}}^{\text{transportation\&production cost}} \right)$$

$$\text{s. t.} \quad \sum_j Y_{ij} \leq x_i \ , \forall i, \qquad (\textit{Capacity constraint})$$

$$\sum_i Y_{ij} \leq d_j \ , \forall j, \quad (\textit{Demand constraint})$$

$$x_i \leq M I_i \ , \forall i, \qquad (\textit{Facility Size constraint})$$

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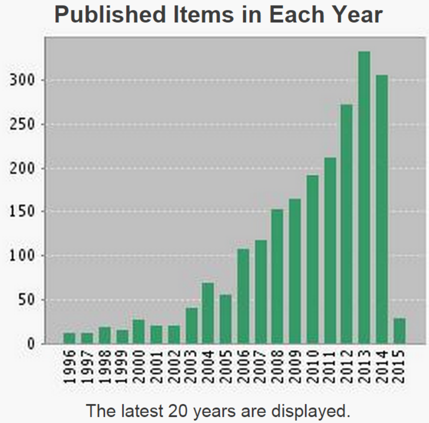
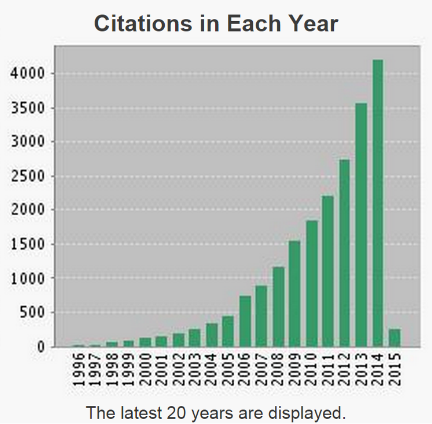
How can one account for demand uncertainty?

# ROBUST OPTIMIZATION IS NOW A WELL ESTABLISHED METHODOLOGY



# A Classical Robust Distribution Problem

- Robust Facility location-transportation model:

$$\begin{aligned} \max_{I\in\{0,1\}^n,x}. \quad & \min_{d\in\mathcal{D}} h(I,x,d) \\ \text{s. t.} \quad & x_i \le MI_i \ , \forall i, \qquad (FacilitySize \ \ constraint) \end{aligned}$$

where $h(I,x,d)$ is the optimal value of

$$\begin{aligned} \max_{Y\ge 0} \quad & \eta \overbrace{\sum_i \sum_j Y_{ij}}^{sold\ product} - \left( \overbrace{c^T x + K^T I}^{location\ cost} + \overbrace{\sum_i \sum_j (p_i + t_{ij}) Y_{ij}}^{transportation\&production\ cost} \right) \\ \text{s. t.} \quad & \sum_j Y_{ij} \le x_i \ , \forall i, \qquad (Capacity \ \ constraint) \\ & \sum_i Y_{ij} \le d_j \, , \forall j, \quad (Demand \ \ constraint) \end{aligned}$$

# Static Robust optimization

- Consider the following static problem:

$$\max_{x \in \mathcal{X}, y} \quad c^T x + f^T y \tag{1a}$$

$$\text{s. t.} \quad Ax + By \leq Dz \ \forall z \in \mathcal{Z} \tag{1b}$$

where we assume $n$ decision variables, $J$ constraints, and $m$ uncertain parameters.

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where we assume $n$ decision variables, $J$ constraints, and $m$ uncertain parameters.

- If $\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, Pz \leq q\}$ is a bounded polyhedral set defined by $K$ constraints, then

$$\text{Problem (1)} \equiv \max_{x\in\mathcal{X},y,\Lambda} \quad c^T x + f^T y$$
$$\text{s. t.} \quad Ax + By + \Lambda q \leq 0$$
$$D + \Lambda P = 0$$
$$\Lambda \geq 0\,,$$

where $\Lambda \in \mathbb{R}^{J\times K}$.

# Two-Stage Robust optimization

- Consider the following two-stage problem:

$$\max_{x \in \mathcal{X}} . \quad \min_{z \in \mathcal{Z}} h(x,z) \tag{2}$$

where

$$\begin{aligned} h(x,z) := \max_{y} . \quad & c^T x + f^T y \\ \text{s. t.} \quad & Ax + By \le Dz\,. \end{aligned}$$

# TWO-STAGE ROBUST OPTIMIZATION

- Consider the following two-stage problem:

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- This problem can be represented as

$$\begin{aligned} \max_{x\in\mathcal{X}, y(\cdot)} \quad & \min_{z\in\mathcal{Z}} c^T x + f^T y(z) \\ \text{s. t.} \quad & Ax + By(z) \le Dz \; \forall z \in \mathcal{Z} \end{aligned}$$

where $y : \mathbb{R}^m \to \mathbb{R}^{n'}$

# COMPLEXITY OF TWO-STAGE ROBUST OPTIMIZATION

- Unfortunately, the two-stage robust optimization problem:

$$\begin{aligned} \max_{x \in \mathcal{X}, y(\cdot)} \quad & \min_{z \in \mathcal{Z}} c^T x + f^T y(z) \\ \text{s. t.} \quad & Ax + By(z) \leq Dz \; \forall z \in \mathcal{Z} \end{aligned}$$

is known to be intractable in general (see Ben-Tal et al. 2004 for a proof).

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  is known to be intractable in general (see Ben-Tal et al. 2004 for a proof).
- Some exact methods have been proposed but without any guarantees about convergence time (see Zeng & Zhao (2013) for a column and constraint generation algorithm)

# AFFINELY ADJUSTABLE ROBUST COUNTERPART (BEN-TAL ET AL. 2004)

- Only consider affine recourse functions :

$$y(z) := y + Yz$$

- The two-stage robust problem reduces to

$$\begin{aligned} \max_{x \in \mathcal{X}, y, Y} . \quad & \min_{z \in \mathcal{Z}} c^T x + f^T (y + Yz) && (3a) \\ \text{s. t.} \quad & Ax + B(y + Yz) \leq Dz \ \forall z \in \mathcal{Z}, && (3b) \end{aligned}$$

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- This model can be reformulated as

$$\begin{aligned} \max_{x \in \mathcal{X}, y, Y, \lambda, \Lambda} . \quad & c^T x + f^T y - q^T \lambda && \text{(4a)} \\ \text{s. t.} \quad & Y^T f + P^T \lambda = 0 && \text{(4b)} \\ & Ax + By + \Lambda q \leq 0 && \text{(4c)} \\ & D - BY + \Lambda P = 0 && \text{(4d)} \\ & \Lambda \geq 0 \,, \lambda \geq 0 && \text{(4e)} \end{aligned}$$

# Lifted AARC (Chen & Zhang 2009)

- Consider affine policies on lifted space:

$$y(b) := y + Y^+z^+ + Y^-z^-$$

where $z = z^+ - z^-$ and $(z^+, z^-) \in \mathcal{Z}'$ with

$$\mathcal{Z}' := CvxHull\left(\{(z^+, z^-) \,|\, \exists z \in \mathcal{Z}, z^+ = \max(0; z),\ z^- = \max(0; -z)\}\right)$$

- The two-stage robust problem reduces to

$$\begin{aligned} \max_{x \in \mathcal{X}, y, Y^+, Y^-} \quad & c^T x + \min_{(z^+, z^-) \in \mathcal{Z}'} f^T(y + Y^+z^+ + Y^-z^-) \\ \text{s. t.} \quad & Ax + B(y + Y^+z^+ + Y^-z^-) \leq D(z^+ - z^-)\ \forall (z^+, z^-) \in \mathcal{Z}' . \end{aligned}$$

# LIFTED AARC (CHEN & ZHANG 2009)

- Consider affine policies on lifted space:

$$y(b) := y + Y^{+}z^{+} + Y^{-}z^{-}$$

where $z = z^{+} - z^{-}$ and $(z^{+}, z^{-}) \in \mathcal{Z}'$ with

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- The two-stage robust problem reduces to

$$\begin{aligned} \max_{x \in \mathcal{X}, y, Y^{+}, Y^{-}}. \quad & c^T x + \min_{(z^{+}, z^{-}) \in \mathcal{Z}'} f^T(y + Y^{+}z^{+} + Y^{-}z^{-}) \\ \text{s. t.} \quad & Ax + B(y + Y^{+}z^{+} + Y^{-}z^{-}) \leq D(z^{+} - z^{-}) \ \forall (z^{+}, z^{-}) \in \mathcal{Z}' . \end{aligned}$$

- We omit to present the reformulation...

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# LINEARIZED ROBUST COUNTERPART (LINRC)

Let our robust optimization problem take the form

$$\max_{x \in \mathcal{X}}.\ \psi(x)\ ,$$

where

$$\psi(x) := \min_{z \in \mathcal{Z}} \max_{y} \quad c^T x + f^T y \tag{5a}$$

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$$\Updownarrow$$

$$\begin{aligned} \psi(x) \ = \ \min_{z \in \mathcal{Z}, \lambda \geq 0} \quad & c^T x + z^T D\lambda - (Ax)^T \lambda \\ & B^T \lambda = f \end{aligned}$$

# LINEARIZED ROBUST COUNTERPART (LINRC)

$$\begin{aligned} \psi(x) \ = \ \min_{z,\lambda \geq 0} \quad & c^T x + trace(D\lambda z^T) - (Ax)^T \lambda \\ \text{s. t.} \quad & Pz \leq q \\ & B^T \lambda = f \end{aligned}$$

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& Pz\lambda^T \leq q\lambda^T \\
& B^T\lambda z^T = fz^T
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\psi(x) \;\geq\; \min_{z,\lambda\geq 0,\Delta} \quad & c^T x + trace(D\Delta) - (Ax)^T\lambda \\
\text{s. t.} \quad & Pz \leq q \\
& B^T\lambda = f \\
& P\Delta^T \leq q\lambda^T \\
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# LINEARIZED ROBUST COUNTERPART (LINRC)

- The following Linearized Robust Counterpart constitute a tractable conservative approximation to the Two-stage Robust problem:

$$\begin{aligned}
\max_{x \in \mathcal{X}} \quad \min_{z,\lambda \geq 0,\Delta} \quad & c^T x + trace(D\Delta) - (Ax)^T \lambda \\
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\end{aligned}
$$

- We omit to provide the reformulation...
- ... actually it just so happens that it is exactly equivalent to the AARC reformulation

# MCCORMICK ENVELOP IMPROVEMENT (MCE-LINRC)

- Assuming that $\mathcal{Z} \subseteq [0,1]^m$ and $\lambda_i^* \subseteq [0, u_i]\ \forall i$, these can help:

$$\lambda_i z_j \leq u_i z_j\ \forall i, j$$
$$\lambda_i z_j = \lambda_i - (1 - z_j)\lambda_i \geq \lambda_i - (1 - z_j)u_i\ \forall i, j$$

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- The problem becomes (MCE-LinRC):

$$\begin{aligned} \max_{x \in \mathcal{X}} \quad \min_{z, \lambda \geq 0, \Delta} \quad & c^T x + trace(D\Delta) - (Ax)^T \lambda \\ \text{s. t.} \quad & Pz \leq q \ \ , \ B^T \lambda = f \\ & P\Delta^T \leq q\lambda^T \ \ , \ B^T \Delta = fz^T \\ & \Delta \leq uz^T \\ & \Delta \geq \lambda e_m - u(e_m - z)^T \end{aligned}$$

- We omit to provide the reformulation...

# SDP based Robust Counterpart (SDP-LinRC)

- Assuming that $\mathcal{Z} \subseteq [0,1]^m$ and $\lambda_i^* \subseteq [0, u_i]\,\forall i$, SDP can help tighten the relaxation

$$\begin{bmatrix} \lambda\lambda^T & \Delta \\ \Delta^T & zz^T \end{bmatrix} \succeq \begin{bmatrix} \lambda \\ z \end{bmatrix} \begin{bmatrix} \lambda^T & z^T \end{bmatrix} \Rightarrow \begin{bmatrix} \Lambda & \Delta & \lambda \\ \Delta^T & Z & z \\ \lambda^T & z^T & 1 \end{bmatrix} \succeq 0$$

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- The problem becomes (SDP-LinRC):

$$\begin{aligned} \max_{x\in\mathcal{X}} \quad \min_{z,\lambda\geq 0,\Delta,\Lambda,Z} \quad & c^T x + trace(D\Delta) - (Ax)^T\lambda \\ \text{s. t.} \quad & Pz \leq q \ , \ B^T\lambda = f \\ & P\Delta^T \leq q\lambda^T \ , \ B^T\Delta = fz^T \\ & \begin{bmatrix} \Lambda & \Delta & \lambda \\ \Delta^T & Z & z \\ \lambda^T & z^T & 1 \end{bmatrix} \succeq 0 \\ & trace(Z) \leq z \ , \ trace(\Lambda) \leq trace(u\lambda^T) \end{aligned}$$

- We omit to provide the reformulation...

# THINGS TO KNOW ABOUT LINRC

1. LinRC is equivalent to AARC.
2. If we replace $z = z^+ - z^-$ and $\mathcal{Z}$ by $\mathcal{Z}'$, then LinRC is equivalent to Lifted AARC.
3. LinRC can be improved on using linear/conic valid inequalities.
   - Ex: MCE-LinRC for facility location-transportation problem
   - Ex: SDP-RC in Inventory problem
4. Using LinRC, we can find new conditions for optimality of Lifted AARC.
   - Ex: Inventory & newsvendor problems

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# Robust Facility Location-Transportation Problem

$$\begin{aligned} \max_{I,x}. \quad & \min_{z\in\mathcal{Z}} \; h(I,x,z) \\ \text{s. t.} \quad & x_i \le M I_i, \; \forall i, \; I_i \in \{0,1\} \; \forall i. \end{aligned}$$

where $\mathcal{Z}$ captures demand uncertainty and $h(I,x,z)$ is the achieved net profit defined as

$$\begin{aligned} h(I,x,z) = \max_{Y}. \quad & -(c^T x + K^T I) + \sum_i \sum_j (\eta - p_i - t_{ij}) Y_{ij} \\ \text{s. t.} \quad & \sum_i Y_{ij} \le \bar{d}_j + \hat{d}_j z_j \, , \forall j, \\ & \sum_j Y_{ij} \le x_i \, , \forall i, \\ & Y_{ij} \ge 0 \, , \forall i, \forall j. \end{aligned}$$

# MCE-LINRC VS. LIFTED AARC

100 trials, 10 potential points for facilities, 10 customers.
$\mathcal{Z}(\Gamma) = \{z \| \|z\|_\infty \leq 1, \|z\|_1 \leq \Gamma\}$ where $\Gamma = 1$ to 10.

| | Lifted AARC | MCE-LinRC |
|---|---|---|
| A | 86% | 88% |
| Max gap | 13% | 6% |
| Time (Sec) | 3.09 | 3.72 |

A: Percentage of trials with optimal solution

# Inspiration for a Penalty Based Approximation Method

- Using MCE-LinRC is equivalent to adding penalized excess variables to uncertain constraint in Lifted AARC.

$$
\begin{aligned}
h(I,x,z) := \max_{Y,\theta}. \quad & -(c^T x + K^T I) + \sum_i \sum_j (\eta - p_i - t_{ij}) Y_{ij} - \sum_j u_j \theta_j \\
\text{s. t.} \quad & \sum_i Y_{ij} \leq \bar{d}_j + \hat{d}_j z_j + \theta_j \,, \forall j \\
& \theta_j \geq 0 \,, \forall j \,,
\end{aligned}
$$

and using a lifted affine decision rule $\theta_j := S_j^+ z_j^+ + S_j^- z_j^- \ \forall j$

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# Minimizing the max of sum of piecewise linear convex functions

Many inventory management problems take the form:

$$\min_{x \in \mathcal{X}}. \quad \max_{z \in \mathcal{Z}} \; h(x,z) \ ,$$

where $h(x,z)$ is piecewise-linear convex function in both $x$ and $z$ variables.

$$h(x,z) = \sum_i \max_k (c_{i,k}(x)^T z + d_{i,k}(x)) \ .$$

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where $h(x,z)$ is piecewise-linear convex function in both $x$ and $z$ variables.

$$h(x,z)=\sum_i \max_k (c_{i,k}(x)^T z + d_{i,k}(x))\;.$$

In fact, one can easily express $h(x,z)$ as a second stage problem

$$\begin{aligned} h(x,z) := \min_y \quad & \sum_i y_i \\ \text{s. t.} \quad & y_i \geq c_{i,k}(x)^T z + d_{i,k}(x)\;, \forall i,k \end{aligned}$$

# Revisiting LinRC Derivations with $\Lambda^+ := z^+\lambda_{i,k}$

$$\psi(x) = \max_{(z^+,z^-)\in\mathcal{Z}',\lambda\geq 0} \quad \sum_{i,k} \lambda_{i,k}(c_{i,k}(x)^T(z^+ - z^-) + d_{i,k}(x))$$
$$\text{s. t.} \quad \sum_k \lambda_{i,k} = 1,\ \forall i$$

which leads to following when using budgeted uncertainty set

$$\psi(x) \leq \max_{\substack{z^+\geq 0, z^-\geq 0,\\ \Lambda^+\geq 0, \Lambda^-\geq 0, \lambda\geq 0}} \quad \sum_{i,k} c_{i,k}(x)^T(\Lambda^+_{i,k} - \Lambda^-_{i,k}) + d_{i,k}(x)\lambda_{i,k}$$
$$\text{s. t.} \quad \sum_k \lambda_{i,k} = 1,\ \forall i$$
$$z^+ + z^- \leq 1\ ,\ e_m^T(z^+ + z^-) = \Gamma$$
$$\sum_k \Lambda^+_{i,k} = z^+\ ,\ \sum_k \Lambda^-_{i,k} = z^-,\ \forall i$$
$$\Lambda^+_{i,k} + \Lambda^-_{i,k} \leq \lambda_{i,k}\ ,\ e_m^T(\Lambda^+_{i,k} + \Lambda^-_{i,k}) = \Gamma\lambda_{i,k},\ \forall i,k$$

# EXAMPLE 1: ROBUST INVENTORY PROBLEM

$$\min_{u} \max_{d \in \mathcal{D}} \quad \sum_{t=1}^{T} c_t u_t + K_t 1_{\{u_t > 0\}} + \max \left\{ \begin{array}{c} h_t(x_1 + \sum_{j=1}^{t}(u_j - d_j)), \\ -p_t(x_1 + \sum_{j=1}^{t}(u_j - d_j)) \end{array} \right\}$$

- Objective: Minimizing ordering and shortage/holding cost
- Decision variable ($u_t$): Stock ordered at the beginning of the $t$th period,
- Uncertainty ($d_t$): Demand during the $t$th period.
- Uncertainty characterization : $d_t := \bar{d}_t + \hat{d}_t z$ with $z \in \mathcal{Z}(\Gamma)$, the budgeted uncertainty

# SDP-LinRC vs. Lifted AARC

Comparison of average performance over a set of 1000 randomly generated inventory problem instances with 10 periods.

| | | Method | |
|---|---|---|---|
| $\Gamma$ | Interval | Lifted AARC | SDP-LinRC |
| 3 | $\leq$0.0001% (optimal) | 52.6% | 56.6% |
| | $\leq$1% (near optimal) | 90.4% | 98.1% |
| | Maximum Gap | 4.6% | 2.0% |
| 5 | $\leq$0.0001% (optimal) | 57.3% | 63.5% |
| | $\leq$1% (near optimal) | 96.6% | 99.70% |
| | Maximum Gap | 2.6% | 1.3% |

# SDP-LinRC vs. Lifted AARC

Comparison of average performance over a set of 1000 randomly generated inventory problem instances with 10 periods.

| | | Method | |
|---|---|---|---|
| $\Gamma$ | Interval | Lifted AARC | SDP-LinRC |
| 3 | ≤0.0001% | 52.6% | 56.6% |
| | ≤1% | 90.4% | 98.1% |
| | Maximum Gap | 4.6% | 2.0% |
| 5 | ≤0.0001% | 57.3% | 63.5% |
| | ≤1% | 96.6% | 99.70% |
| | Maximum Gap | 2.6% | 1.3% |

# SDP-LINRC VS. LIFTED AARC

Comparison of average performance over a set of 1000 randomly generated inventory problem instances with 10 periods.

| | | Method | |
|---|---|---|---|
| $\Gamma$ | Interval | Lifted AARC | SDP-LinRC |
| 3 | $\leq$0.0001% | 52.6% | 56.6% |
| | $\leq$1% | 90.4% | 98.1% |
| | Maximum Gap | 4.6% | 2.0% |
| 5 | $\leq$0.0001% | 57.3% | 63.5% |
| | $\leq$1% | 96.6% | 99.70% |
| | Maximum Gap | 2.6% | 1.3% |

# Robust Multi-item Newsvendor

- The following robust multi-item newsvendor problem can be solved optimally using a LinRC model

$$\max_{x\in\mathcal{X}} \min_{\zeta\in\mathcal{Z}(\Gamma)} \sum_i (r_i-c_i)x_i - \max\left\{ \begin{array}{c} (r_i-s_i)(x_i-\bar{d}_i-\hat{d}_i z_i)\,, \\ p_i(\bar{d}_i+\hat{d}_i z_i - x_i) \end{array} \right\}$$

when using the budgeted uncertainty set with integer budget.

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  when using the budgeted uncertainty set with integer budget.
- Proof relies on fact that vertices of relaxed feasible set for $(z^+, z^-, \Lambda^+, \Lambda^-, \lambda)$ are integer hence the linearisation $\Lambda^+_{i,k} = z^+\lambda_{i,k}$ and $\Lambda^-_{i,k} = z^-\lambda_{i,k}$ are exact due to constraints:

$$\lambda \in [0,1]\,,\ z^+ \in [0,1]^m\,,\ z^- \in [0,1]^m$$
$$\Lambda^+_{i,k} \geq 0\,,\ \Lambda^-_{i,k} \geq 0$$
$$\sum_k \Lambda^+_{i,k} = z^+\,,\ \sum_k \Lambda^-_{i,k} = z^-$$
$$\Lambda^+_{i,k} + \Lambda^-_{i,k} \leq \lambda_{i,k}$$

# DISTRIBUTION FREE MULTI-ITEM NEWSVENDOR

The following distributionally robust problem can also be solved optimally using a LinRC

$$\min_{x \in \mathcal{X}} \max_{F \in \mathcal{D}} \mathbb{E}_F \left[ \sum_i (r_i - c_i)x_i - \max \left\{ \begin{array}{c} (r_i - s_i)(x_i - \bar{d}_i - \hat{d}_i z_i) , \\ p_i(\bar{d}_i + \hat{d}_i z_i - x_i) \end{array} \right\} \right] ,$$

where

$$\mathcal{D} = \left\{ F \in \mathcal{M} \,\middle|\, \begin{array}{l} \mathbb{P}_F(z \in \mathcal{Z}(\Gamma)) = 1 \\ \mathbb{E}_F[z] = \mu \\ \mathbb{E}_F[(z - \mu)^+] \leq \boldsymbol{r}^+ \\ \mathbb{E}_F[(\mu - z)^+] \leq \boldsymbol{r}^- \end{array} \right\} .$$

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# CONCLUSION

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2. LinRC can be improved by using linear/conic valid inequalities
   - Ex: Facility location-transportation problem (MCE-LinRC$\equiv$ Lifted AARC+penalty method)
   - Ex: Multi-period inventory problem (SDP-LinRC is tighter)

# CONCLUSION

1. Conservative tractable approximations of two-stage robust optimization can be obtained using the LinRC approach
   - A simple application of LinRC is equivalent to Lifted AARC
2. LinRC can be improved by using linear/conic valid inequalities
   - Ex: Facility location-transportation problem (MCE-LinRC≡ Lifted AARC+penalty method)
   - Ex: Multi-period inventory problem (SDP-LinRC is tighter)
3. LinRC help identify optimality conditions of Lifted AARC
   - Ex: Inventory problem with simplex uncertainty set
   - Ex: Multi-item newsvendor problem with budgeted uncertainty set
   - Ex: Distribution free Multi-item newsvendor problem information limited to budget, mean and first order partial moment.

Questions & Comments ...

... Thank you!