Addressing Model Ambiguity in the Expected Utility Framework

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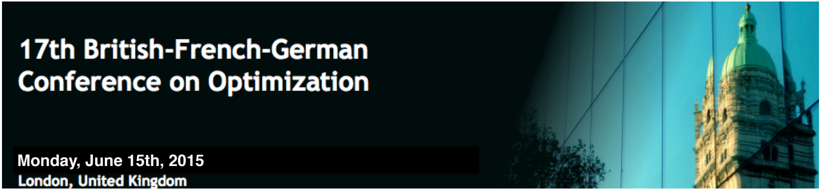
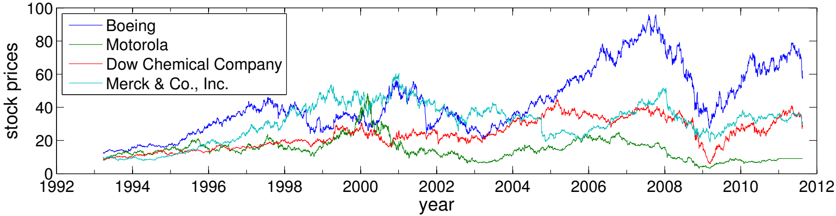
Joint work with Yinyu Ye (Stanford) and Benjamin Armbruster (Northwestern)



The Portfolio Selection Problem



- An individual meets with his financial advisor to tell him he wishes to invest in a given industrial sector, country, etc.
- Since uncertain factors affect performance, a "good" portfolio is one where the risks of losses are best justified by the potential gains
- How can one trade-off optimally the risks and the returns taking into account his own perception of what is a serious risk?

Von Neumann-Morgenstern Expected Utility

If the investor agrees with the following axioms:

1. Completeness : He can order any two gambles
2. Transitivity : $\mathcal{H}_1 \succeq \mathcal{H}_2 \succeq \mathcal{H}_3 \ \Rightarrow \ \mathcal{H}_1 \succeq \mathcal{H}_3$
3. Continuity : If $\mathcal{H}_1 \succeq \mathcal{H}_2 \succeq \mathcal{H}_3$ then there is a p such that $\mathcal{H}_2 \sim p\mathcal{H}_1 + (1-p)\mathcal{H}_3$
4. Independence : If $\mathcal{H}_1 \succeq \mathcal{H}_2$ then $p\mathcal{H}_1 + (1-p)\mathcal{H}_3 \succeq p\mathcal{H}_2 + (1-p)\mathcal{H}_3$ for all p and $\mathcal{H}_3$

then the preference he expresses between any two gambles must be representable by an expected utility measure:

$$\mathcal{H}_1 \succeq \mathcal{H}_2 \quad \Leftrightarrow \quad E[u(\mathcal{H}_1)] \geq E[u(\mathcal{H}_2)]$$

Expected Utility Framework

When applying the expected utility framework to a decision problem:

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \mathbb{E}\left[u(\mathrm{h}(\boldsymbol{x},\boldsymbol{\xi}))\right],$$

where $\boldsymbol{x}$ = decisions, $\boldsymbol{\xi}$ = uncertain parameters, $h(\boldsymbol{x},\boldsymbol{\xi})$ = profit,

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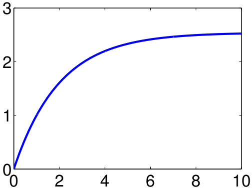
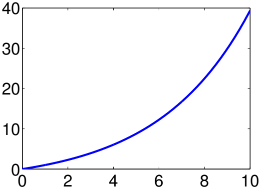
it is assumed that we know:

- The distribution of the random vector $\boldsymbol{\xi}$
- A utility function that matches investor's attitude to risk



Retirement Fund



Casino de Monte-Carlo

Difficulties encountered in practice

Difficulties of developing an accurate probabilistic model:

- Need to collect enough observations
- Need to consult with experts of the field of practice
- Need to make simplifying assumptions
- Unforeseen events (e.g., economic crisis) might occur

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Difficulties of developing an accurate utility function:

- Need to compare a large number of gambles
- Need to accept structural properties
- Perception might be biased

Modern Robust Optimization Framework

Generally attributed to Ben-Tal & Nemirovski (1998), this framework implements a worst-case approach to dealing with model ambiguity.

$$\max_{\boldsymbol{x}\in\mathcal{X}}. \ \{h(\boldsymbol{x},\boldsymbol{y})\}_{\boldsymbol{y}\in\mathcal{Y}} \quad \rightarrow \quad \max_{\boldsymbol{x}\in\mathcal{X}}. \ \inf_{\boldsymbol{y}\in\mathcal{Y}} h(\boldsymbol{x},\boldsymbol{y})$$

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Success of the robust optimization relies in part on applying duality to combine inner and outer problems

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathcal{X}}. \ \min_{\boldsymbol{y}:\boldsymbol{A}\boldsymbol{y}\leq\boldsymbol{b}} \boldsymbol{y}^T\boldsymbol{x} &\equiv \max_{\boldsymbol{x}\in\mathcal{X}}. \ \min_{\boldsymbol{y}} \ \max_{\lambda\geq 0} \ \boldsymbol{y}^T\boldsymbol{x} + \lambda^T(\boldsymbol{A}\boldsymbol{y}-\boldsymbol{b}) \\ &\equiv \max_{\boldsymbol{x}\in\mathcal{X}}. \ \max_{\lambda\geq 0} \ \min_{\boldsymbol{y}} \ \boldsymbol{y}^T\boldsymbol{x} + \lambda^T(\boldsymbol{A}\boldsymbol{y}-\boldsymbol{b}) \\ &\equiv \max_{\boldsymbol{x}\in\mathcal{X},\lambda\geq 0}. \ -\boldsymbol{b}^T\lambda \ \text{ s.t. } \ \boldsymbol{x}+\boldsymbol{A}^T\lambda = 0 \end{aligned}$$

Robust Expected Utility

In this talk, we investigate how to address model ambiguity in the expected utility framework using robust optimization

- Distributionally robust optimization

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \; \inf_{F\in\mathcal{D}} \mathbb{E}_F[u(\mathsf{h}(\boldsymbol{x},\boldsymbol{\xi}))]$$

- Preference robust optimization

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \; \text{“} \inf_{u\in\mathcal{U}} \mathbb{E}_F[u(\mathsf{h}(\boldsymbol{x},\boldsymbol{\xi}))] \text{”}$$

Outline

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Dealing with model ambiguity: Ellsberg Paradox

Consider an urn with 30 blue balls and 60 other balls that are either red or green (you don't know how many are red or green).

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Experiment 2: Choose among the following two gambles

- Gamble C: If you draw blue or green ball, then you win 100$
- Gamble D: If you draw red or green ball, then you win 100$

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- Gamble D: If you draw red or green ball, then you win 100$

If you clearly prefer Gamble A & D, then you are averse to model ambiguity

Distributionally Robust Optimization

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- Distributionally Robust Optimization values the lowest performing possible stochastic model

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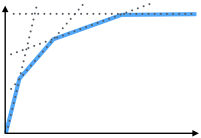
- Important milestones:
  - 1958: H. Scarf introduces DRO
  - 1989: I. Gilboa et al. introduces maxmin expected utility
  - 2007: I. Popescu solves μ-Σ portfolio prob.
  - 2010: Bertsimas et al. solves linear μ-Σ prob.
  - 2010: Goh et al. develops library for Matlab
  - 2010: Delage et al. solves concave-convexe $\mathcal{S}$-$\tilde{\mu}$-$\tilde{\Sigma}_{\max}$ prob.
  - 2014: Wiesemann et al. solves or approximates conic prob.

A Classic Reduction I

Let's make three assumptions about $\mathbb{E}\left[u(\mathrm{h}(\boldsymbol{x},\boldsymbol{\xi}))\right]$.

1. The profit function is concave in $\boldsymbol{x}$ and convex in $\boldsymbol{\xi}$
   - In portfolio optimization, $h(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{\xi}^T\boldsymbol{x}$
2. The utility function is piecewise linear concave :



$$u(y) = \min_{1\leq k\leq K} \quad a_k y + b_k$$

3. The information about F is captured by

$$\mathcal{D}(\gamma) = \left\{ F \left| \begin{array}{l} \mathbb{P}(\boldsymbol{\xi} \in \mathcal{S}) = 1 \\ \|\mathbb{E}[\boldsymbol{\xi}] - \hat{\mu}\|^2_{\hat{\Sigma}^{-1/2}} \leq \gamma_1 \\ \mathbb{E}\left[(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})(\boldsymbol{\xi} - \hat{\boldsymbol{\mu}})^{\mathsf{T}}\right] \preceq (1+\gamma_2)\hat{\Sigma} \end{array} \right. \right\}$$

A Classic Reduction II

- The DRO problem with $\mathcal{D}(\gamma)$ is equivalent to

$$\begin{array}{lll} \max_{\boldsymbol{x}\in\mathcal{X}} & \inf_{F} & \mathbb{E}_F[u(\mathsf{h}(\boldsymbol{x},\boldsymbol{\xi}))] \\ & \text{s.t.} & \mathbb{P}_F(\boldsymbol{\xi}\in\mathcal{S}) = 1 \\ & & \|\mathbb{E}_F[\boldsymbol{\xi}] - \hat{\mu}\|^2_{\hat{\Sigma}^{-1/2}} \leq \gamma_1 \\ & & \mathbb{E}_F[(\boldsymbol{\xi}-\hat{\boldsymbol{\mu}})(\boldsymbol{\xi}-\hat{\boldsymbol{\mu}})^{\mathsf{T}}] \preceq (1+\gamma_2)\hat{\Sigma} \end{array}$$

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A Classic Reduction II

- The DRO problem with $\mathcal{D}(\gamma)$ is equivalent to

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathcal{X}}. \quad \max_{\mathbf{Q},\mathbf{q},r} \quad & r - \left(\gamma_2\hat{\Sigma} + \hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\mathsf{T}\right) \bullet \mathbf{Q} - \hat{\boldsymbol{\mu}}^\mathsf{T}\mathbf{q} - \sqrt{\gamma_1}\,\|\hat{\Sigma}^{1/2}(\mathbf{q} + 2\mathbf{Q}\hat{\boldsymbol{\mu}})\| \\ \text{s.t.} \quad & r \leq u(\mathrm{h}(\boldsymbol{x},\boldsymbol{\xi})) + \boldsymbol{\xi}^\mathsf{T}\mathbf{q} + \boldsymbol{\xi}^\mathsf{T}\mathbf{Q}\boldsymbol{\xi} \; \forall\, \boldsymbol{\xi} \in \mathcal{S} \\ & \mathbf{Q} \succeq 0 \end{aligned}$$

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- For portfolio selection, if $\mathcal{S}$ = polyhedron or ellipsoid, then DRO equivalent to semi-definite program.
  E.g., when $\mathcal{S} = \mathbb{R}^m$, constraint $(\star)$ can be replaced by

$$\begin{bmatrix} \mathbf{Q} & (\mathbf{q} + a_k\boldsymbol{x})/2 \\ (\mathbf{q} + a_k\boldsymbol{x})^{\mathsf{T}}/2 & b_k - r \end{bmatrix} \succeq 0\ ,\ \forall\, k$$

Distributionally Robust Portfolio Optimization

Let's consider the case of portfolio optimization:

$$\max_{\boldsymbol{x}\in\mathcal{X}}. \ \min_{F\in\mathcal{D}} \mathbb{E}_F[u(\boldsymbol{\xi}^\mathsf{T}\boldsymbol{x})] \ \ ,$$

where x_i is how much is invested in stock i with future return ξ_i.

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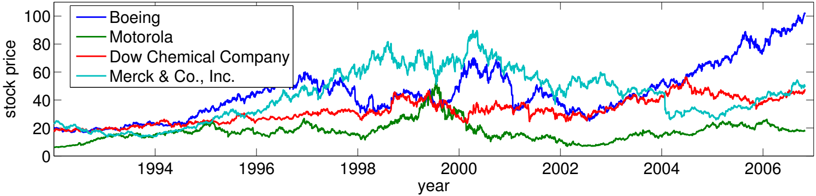
Does the robust solution perform better than solution of expected utility problem with fixed $\hat{F}$?

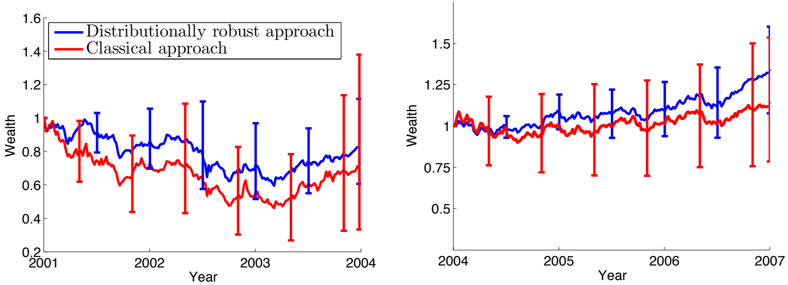
$$\mathcal{D} = \mathcal{D}(\gamma) \qquad \text{vs.} \qquad \mathcal{D} = \{\hat{F}\}$$

Experiments in Portfolio Optimization

30 stocks tracked over years 1992-2007 using Yahoo! Finance



Wealth Evolution for 300 Experiments



- 10% and 90% percentiles are indicated periodically.

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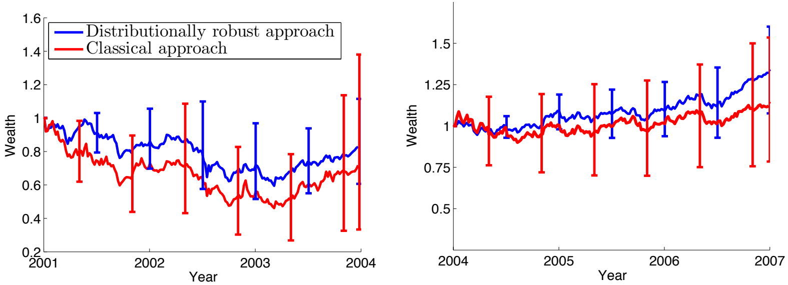

- 10% and 90% percentiles are indicated periodically.
- 79% of time, the DRO outperformed the classical approach
- 67% improvement on average using DRO with $\mathcal{D}(\gamma)$

Other applications

- Territory partitioning for multi-vehicle routing problem with J. G. Carlsson ▸ Details
- Fleet mix optimization problem with S. Arroyo and Y. Ye ▸ Details
- Quadratic knapsack problem with J. Cheng and A. Lisser
- Multi-item newsvendor problem with A. Ardestani-Jaafari
- Many others in scheduling, environmental policies, smart grid management, etc.

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How does One Assesses Risk Tolerance ?

Here are two questions from a survey proposed by [Grable & Lytton, Financial Services Review (1999)].

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- You are on a TV game show and can choose one of the following. Which would you take?
  1. $1,000 in cash
  2. A 50% chance at winning $ 5000
  3. A 25% chance at winning $ 10,000
  4. A 5% chance at winning $100,000

Common Utility Function Estimation Techniques I

Parametric approach:

- One assumes that the function has a specific parametric form
  - Negative exponential utility (CARA)
  - Power utility (CRRA)
  - HARA (incr/decreasing absolute/relative risk aversion)

$$u(y) = \frac{1-\eta}{\eta}\left(\frac{ay}{1-\eta} + b\right)^{\eta}$$

- Ask enough questions to identify the parameters

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- Why it might fail ?
  - Investor must commit to global structure
  - Ignores how confident we are in the final choice of parameters

Common Utility Function Estimation Techniques II

Non-parametric approach:

- Identify the certainty equivalents of a list of lotteries. By answering:
  - What is the smallest amount c_i of money you would take instead of playing a lottery $\mathcal{L}_i$?
- Find a piecewise linear utility function that satisfies these certainty equivalents

$$E[u(\mathcal{L}_i)] = u(c_i) \ \ \forall i$$

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- Why it might fail ?
  - Expresses risk neutrality between breakpoints
  - Ignores how confident we are in the final choice of $u(\cdot)$

Applying the Robust Optimization Framework

Information can be used to characterize a set $\mathcal{U}$ of plausible utility functions. I.e., any $u(\cdot)$ such that:

- Global info:
  - Risk aversion : $u(\cdot)$ is concave
  - Prudence : $u'(\cdot)$ is convex
  - S-shaped : $u(\cdot)$ convex-concave
- Local info :

$$E[u(\mathcal{W}_k)] \geq E[u(\mathcal{Y}_k)] \;\; \forall k$$

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Unfortunately, a direct application of RO is meaningless

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \;\; \inf_{u\in\mathcal{U}} \mathbb{E}\left[u(\mathrm{h}(\boldsymbol{x},\boldsymbol{\xi}))\right] \;\; = \;\; -\infty$$

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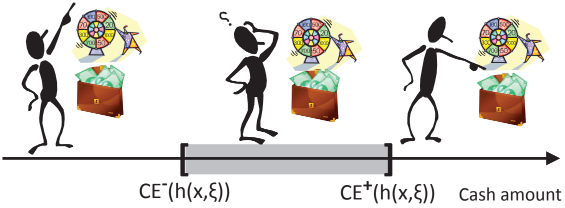
Even if we force $u(0) = 0$ $u(1) = 1$, model promotes risk neutrality.

Ambiguity about the Certainty Equivalent

- Given a decision $\boldsymbol{x}$, we can start by defining an interval $[CE^{-}(h(x,\xi)), CE^{+}(h(\boldsymbol{x},\xi))]$ of plausible minimum certain return that decision maker would prefer to random profit $h(\boldsymbol{x},\xi)$



Robust Certainty Equivalent Approach

Theorem

Identifying the decision that maximizes the lowest perceived CE

$$\max_{\boldsymbol{x}\in\mathcal{X}}. \quad CE^{-}(h(x,\boldsymbol{\xi}))\,, \quad a.k.a. \quad \max_{\boldsymbol{x}\in\mathcal{X}}. \quad \inf_{u\in\mathcal{U}} CE_u(h(x,\xi))\,,$$

can be done efficiently.

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Proof:

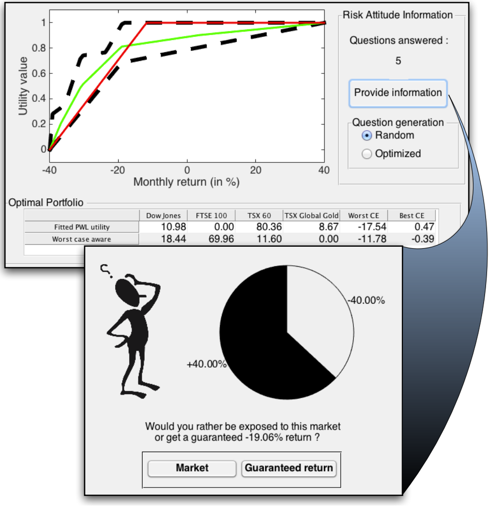
- Objective is quasiconcave and reduces to

$$\max_{\boldsymbol{x},t} \; t \;\; \text{s.t.} \;\; CE_u(h(\boldsymbol{x},\boldsymbol{\xi})) \geq t \;\; \forall\, u \in \mathcal{U}$$

or equiv. $\max_{t} \; t \;\; \text{s.t.} \;\; \max_{\boldsymbol{x}\in\mathcal{X}} \inf_{u\in\mathcal{U}} E[u(h(\boldsymbol{x},\boldsymbol{\xi}))] - u(t) \geq 0$

- Infimum over $u \in \mathcal{U}$ can be reduced to finite dimensional program so that duality can be applied details

A Tool for Interacting with Investors



Numerical experiments

- Obtained from Yahoo! Finance historical stock returns for 350 companies from 1993 to 2011
- Ran extensive amount of trials using last 50 weekly returns to decide investment among 10 assets for next week
- In each experiment, the investor has an unknown risk averse utility function and compares up to 80 pairs of gambles

Experimental Results

Function optimized	Portfolio's true CE value (in perc. point)		
	5 questions	20 questions	80 questions
Exponential	-0.05	-0.12	**-0.13**
Fitted PWL	**-0.60**	-0.11	0.05
Worst-case	-0.14	-0.08	0.06
Worst-case prudent	-0.13	-0.05	0.08
True « $\frac{-u''(y)y}{u'(y)} = \frac{20}{y}$ »	0.12	0.12	0.12

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Observations:

- Using wrong utility function can mislead the choice of portfolio
- Robust approach makes good use of available preference info
- Quality of portfolio is improved as more information is provided

Accounting for elicitation errors

- Since "to err is human", we should account for mislabelling of the compared lotteries
- Hence, that for some rand perception noise ϵ, we have that

$$E[u(W_k)] + \epsilon_k \geq E[u(Y_k)]$$

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$$E[u(W_k)] + \epsilon_k \geq E[u(Y_k)]$$

- In that case, one could consider $u(\cdot)$ plausible as long as $\exists \delta \geq 0$ such that $\sum_k \delta_k \leq \Gamma$ and that

$$E[u(W_k)] + \delta_k \geq E[u(Y_k)] \; \forall \, k$$

- This can easily be incorporated to the model

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Conclusion & Future Work

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Conclusion & Future Work

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  - Disregarding it can be misleading
- Accounting for distribution ambiguity or ambiguity in risk preferences isn't computationally demanding
  - It remains to further study whether both ambiguities can be accounted for jointly (see Haskell et al. 2014)
- Studying the sensitivity of optimal solution with respect to modelled ambiguity can be helpful
  - Value of stochastic modelling
  - Guidance for risk tolerance assessment

Bibliography I

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Questions & Comments ...

... Thank you!

Dealing with model ambiguity: Ellsberg Paradox

Consider an urn with 30 blue balls and 60 other balls that are either red or green (you don't know how many are red or green).

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- Gamble D: If you draw red or green ball, then you win 100$

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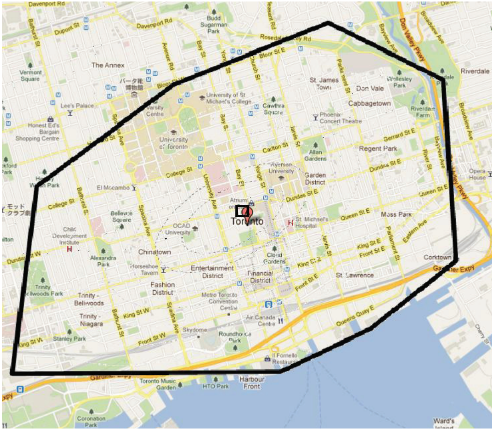
If you clearly prefer Gamble A & D, then you are averse to model ambiguity

Outline

Multi-Vehicle Routing on a Planar Region

- Divide a planar region into K subregions, each serviced by a different vehicle, so that the total workload be most evenly distributed among the fleet



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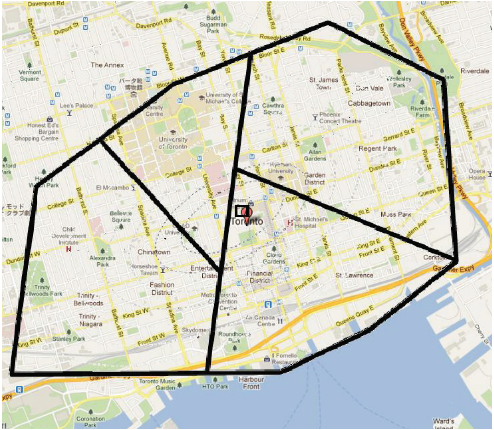

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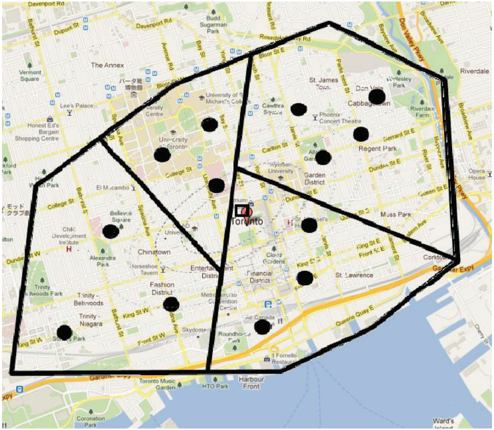

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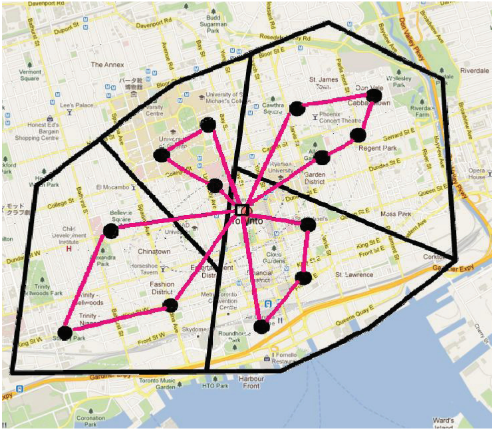

Distributionally Robust Partitioning

- Given $\mathcal{D}$, we partition so that the largest workload over the worst distribution of demand points is as small as possible

$$\min_{\{\mathcal{R}_1,\mathcal{R}_2,...,\mathcal{R}_K\}} \sup_{F\in\mathcal{D}} \left\{ \max_i \mathbb{E}[TSP(\{\xi_1,\xi_2,...,\xi_N\}\cap\mathcal{R}_i)] \right\} ,$$

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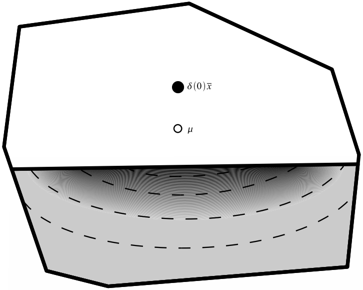
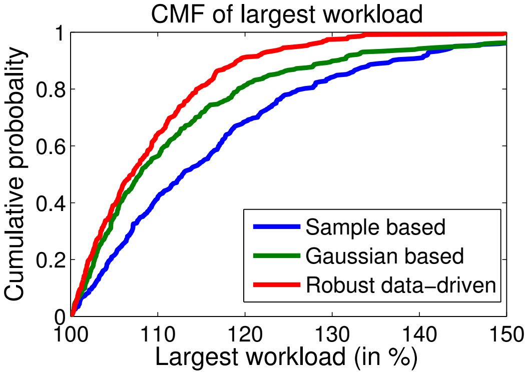
- A side product is to characterize for any partition what is a worst-case distribution of demand locations



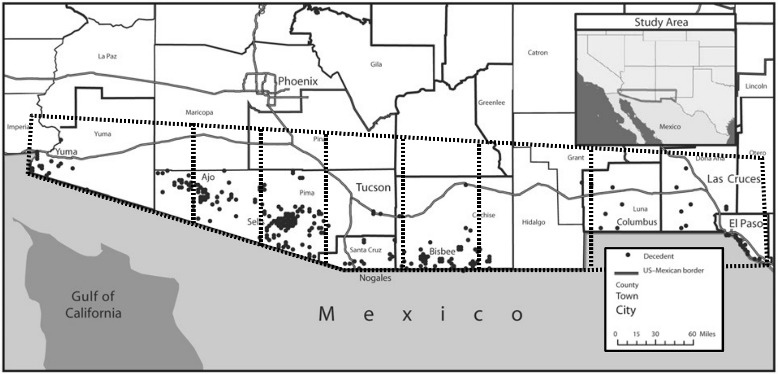
Distributionally Robust Partitioning

We simulated three partition schemes on a set of randomly generated parcel delivery problems where the territory needed to be divided into two regions and the demand is drawn from a mixture of truncated Gaussian distribution



Border Patrol Workload Partitioning

Robust partitions of the USA-Mexico border obtained using our branch & bound algorithm.



▸ Back to DRO applications

Outline

The Robustness of the Deterministic Solution

If we are risk neutral we might not even need distribution information

Theorem

The solution of

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \mathbb{E}[h(\boldsymbol{x},\mu)]$$

is optimal with respect to

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \quad \inf_{F\in\mathcal{D}(\mu,\Psi)} \quad \mathbb{E}_F[h(\boldsymbol{x},\boldsymbol{\xi})] \ ,$$

for any set of convex functions Ψ *with*

$$\mathcal{D}(\mu,\Psi) = \left\{ F \left| \begin{array}{l} \mathbb{E}[\boldsymbol{\xi}] = \mu \\ \mathbb{E}[\psi(\boldsymbol{\xi})] \leq 0 \ , \ \forall \psi \in \Psi \end{array} \right. \right\} .$$

The Value of Stochastic Modelling

Consider the situation:

1. We know of a set $\mathcal{D}$ such that $F \in \mathcal{D}$
2. We have a candidate solution $\boldsymbol{x}_1$ in mind
3. Is it worth developing a stochastic model: $\mathcal{D} \rightarrow F$?
   (a) If yes, then develop a model & solve it
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The Value of Stochastic Modelling ($\mathcal{VSM}$) gives an optimistic estimate of the value of obtaining perfect information about F.

$$\mathcal{VSM}(\boldsymbol{x}_1) := \sup_{F \in \mathcal{D}} \left\{ \max_{\boldsymbol{x}_2} \mathbb{E}_F[\mathrm{h}(\boldsymbol{x}_2, \boldsymbol{\xi})] - \mathbb{E}_F[\mathrm{h}(\boldsymbol{x}_1, \boldsymbol{\xi})] \right\}$$

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Theorem

Unfortunately, evaluating $\mathcal{VSM}(\boldsymbol{x}_1)$ exactly is NP-hard in general.

Bounding the Value of Stochastic Modelling

Theorem

If $\mathcal{S} \subseteq \{\boldsymbol{\xi} \,|\, \|\boldsymbol{\xi}\|_1 \leq \rho\}$, an upper bound can be evaluated in $O(d^{3.5} + d\, T_{DCP})$ using:

$$
\begin{aligned}
\mathcal{UB}(\boldsymbol{x}_1, \bar{\boldsymbol{y}}_1) \;:=\; \min_{s,\boldsymbol{q}} \quad & s + \boldsymbol{\mu}^\top \boldsymbol{q} \\
\text{s.t.} \quad & s \geq \alpha(\rho \boldsymbol{e}_i) - \rho \boldsymbol{e}_i^\top \boldsymbol{q} \;, \forall\, i \in \{1, ..., d\} \\
& s \geq \alpha(-\rho \boldsymbol{e}_i) + \rho \boldsymbol{e}_i^\top \boldsymbol{q} \;, \forall\, i \in \{1, ..., d\} \,,
\end{aligned}
$$

where $\alpha(\boldsymbol{\xi}) = \max_{\boldsymbol{x}_2} h(\boldsymbol{x}_2, \boldsymbol{\xi}) - h(\boldsymbol{x}_1, \boldsymbol{\xi}; \bar{\boldsymbol{y}}_1)$.

Are Airlines Adventurous in their Fleet Acquisition?

- Fleet composition is a difficult decision problem:
  - Fleet contracts are signed 10 to 20 years ahead of schedule.
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- Yet, most airline companies sign these contracts based on a single scenario of what the future may be.
- Are airlines companies being neglectful?

Mathematical formulation for Fleet Mix Problem

The fleet composition problem is a stochastic mixed integer LP

$$\max_{\boldsymbol{x}}.\ \mathbb{E}[-\underbrace{\boldsymbol{o}^\top \boldsymbol{x}}_{\text{ownership cost}} + \underbrace{h(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{c}}, \tilde{\boldsymbol{L}})}_{\text{future profits}}]\ ,$$

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with $h(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{c}}, \tilde{\boldsymbol{L}}) :=$

$$\max_{z \geq 0, y \geq 0, w} \quad \sum_k \Big(\sum_i \overbrace{\tilde{p}_i^k w_i^k}^{\text{flight profit}} - \overbrace{\tilde{c}_k (z_k - x_k)^+}^{\text{rental cost}} + \overbrace{\tilde{L}_k (x_k - z_k)^+}^{\text{lease revenue}} \Big)$$

$$\text{s.t.} \quad w_i^k \in \{0,1\} \;,\; \forall\, k, \forall\, i \quad \& \quad \sum_k w_i^k = 1 \;,\; \forall\, i \quad \Big\} \text{ Cover}$$

$$y^k_{g \in \text{in}(v)} + \sum_{i \in \text{arr}(v)} w_i^k = y^k_{g \in \text{out}(v)} + \sum_{i \in \text{dep}(v)} w_i^k \;,\; \forall\, k, \forall\, v \quad \Big\} \text{ Balance}$$

$$z_k = \sum_{v \in \{v | \text{time}(v) = 0\}} \big(y^k_{g \in \text{in}(v)} + \sum_{i \in \text{arr}(v)} w_i^k\big) \;,\quad \forall k \quad \Big\} \text{ Count}$$

Experiments in Fleet Mix Optimization

We experimented with three test cases :

1. 3 types of aircraft, 84 flights, $\sigma_{\tilde{p}_i}/\mu_{\tilde{p}_i} \in [4\%, 53\%]$
2. 4 types of aircraft, 240 flights, $\sigma_{\tilde{p}_i}/\mu_{\tilde{p}_i} \in [2\%, 20\%]$
3. 13 types of aircraft, 535 flights, $\sigma_{\tilde{p}_i}/\mu_{\tilde{p}_i} \in [2\%, 58\%]$

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Results:

Test cases	CPU Time		DRO sub-optimality	
	DRO	SP with $\hat{F}$	Under $\hat{F}$	$\forall\ F \in \mathcal{D}$
#1	0.6 s	3 min	0.001%	$< 6\%$
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Conclusions:

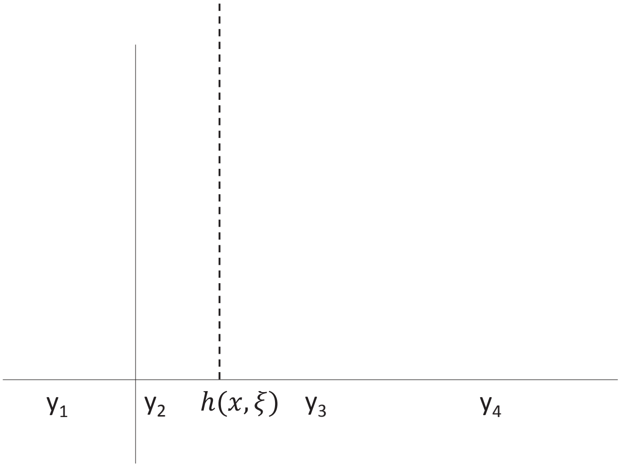
- It's wasteful to invest more than 7% of profits in extra info

Back to DRO applications

Outline

Constructing the Worst-case Utility I

- Define $\mathcal{S} = \{y_1, y_2, ..., y_N\}$ contains support of $\mathcal{W}_k$ and $\mathcal{Y}_k$, and t.
- Define the values $\alpha_i := u(y_i)$



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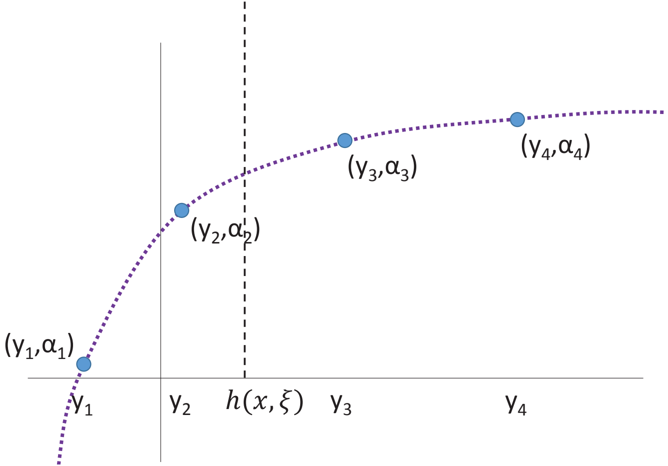

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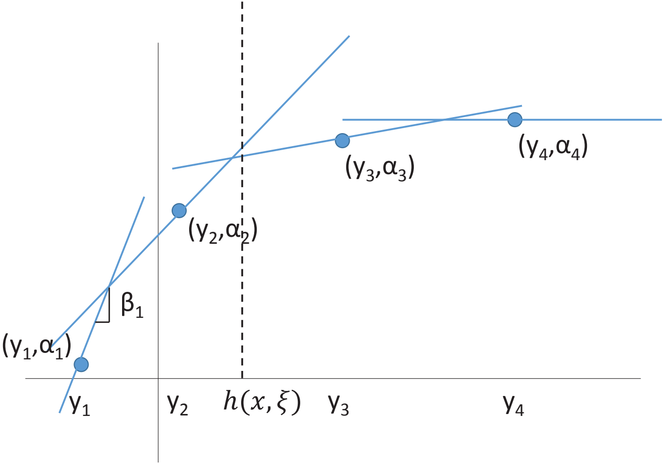
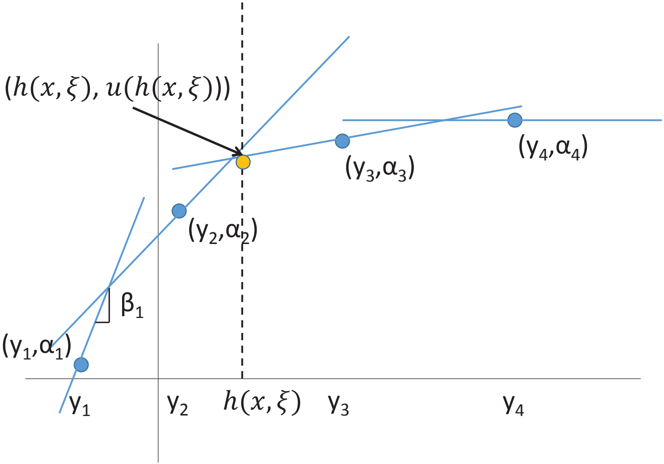


Constructing the Worst-case Utility II

- Once all $(y_i, u(y_i))$ are fixed, identify the worst-case utility value for $u(h(\boldsymbol{x}, \xi))$.



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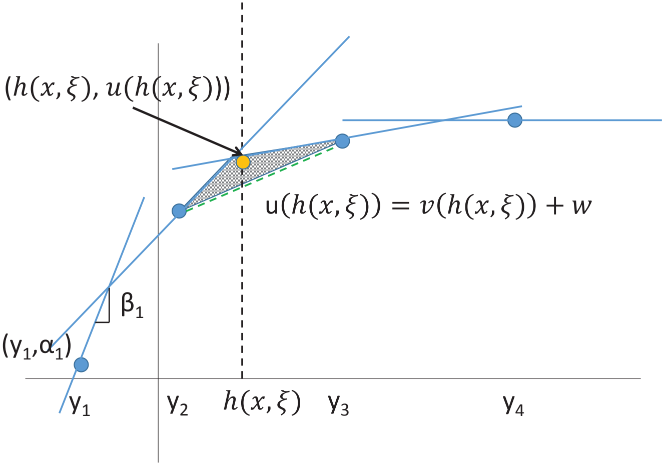

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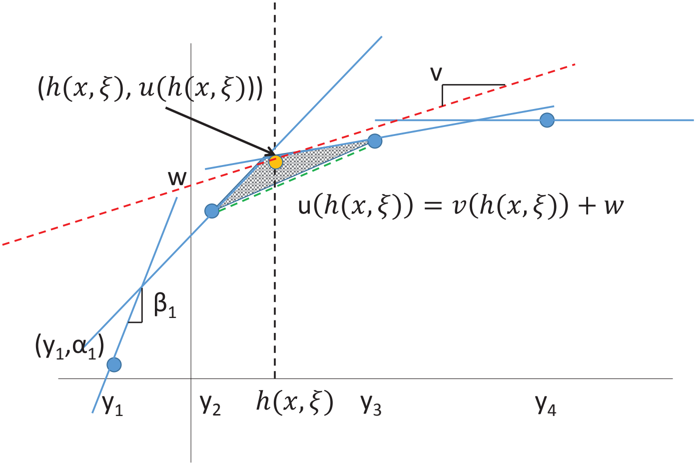

LP reformulation of $\inf_{u \in \mathcal{U}} E[u(h(\boldsymbol{x}, \xi))] - u(t)$

We wish to find an $\boldsymbol{x}$ s.t. the following finite dimensional LP has a positive optimal value:

$$\begin{aligned}
\min_{\alpha,\beta,v,w} \quad & \sum_i p_i(v_i h(\boldsymbol{x}, \xi^i) + w_i) - \alpha_t \\
\text{s.t.} \quad & v_i y_i + w_i \geq \alpha_j \;\; \forall\, i,j \quad \text{(Risk aversion at } h(\boldsymbol{x}, \xi^i)) \\
& \sum_j P(\mathcal{W}_k = y_j)\alpha_j \geq \sum_j P(\mathcal{Y}_k = y_j)\alpha_j \;\; \forall\, k \quad \text{(Local pref's)} \\
& \alpha_{j+1} \leq \alpha_j + \beta_j(y_{j+1} - y_j) \;\; \forall\, j \quad \text{(Risk aversion at } y_j\text{'s)} \\
& \alpha_{j-1} \leq \alpha_j + \beta_j(y_{j-1} - y_j) \;\; \forall\, j \\
& v \geq 0, \beta \geq 0 \quad \text{(Monotonicity)}
\end{aligned}$$

After taking the dual of this LP, we can join the maximization with $\boldsymbol{x} \in \mathcal{X}$

Back to talk