Linear & Conic Programming Reformulations of Two-Stage Robust Linear Programs

Erick Delage
CRC in decision making under uncertainty
Department of Decision Sciences
HEC Montreal

(joint work with Amir Ardestani-Jaafari)
(special thanks to Samuel Burer)

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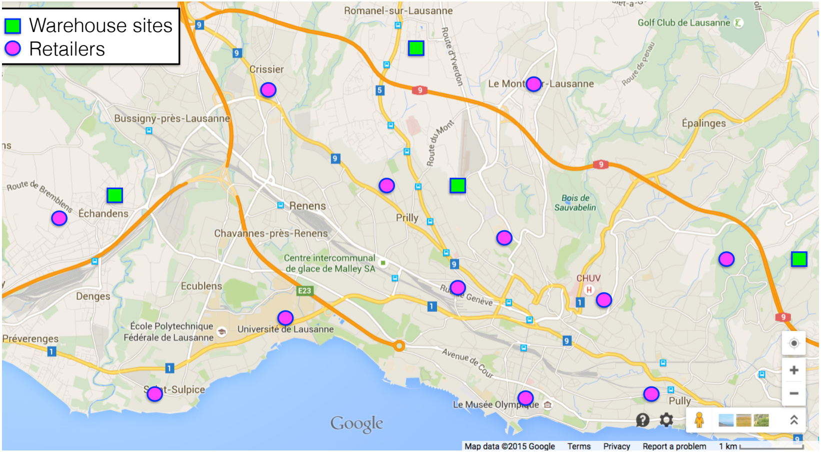
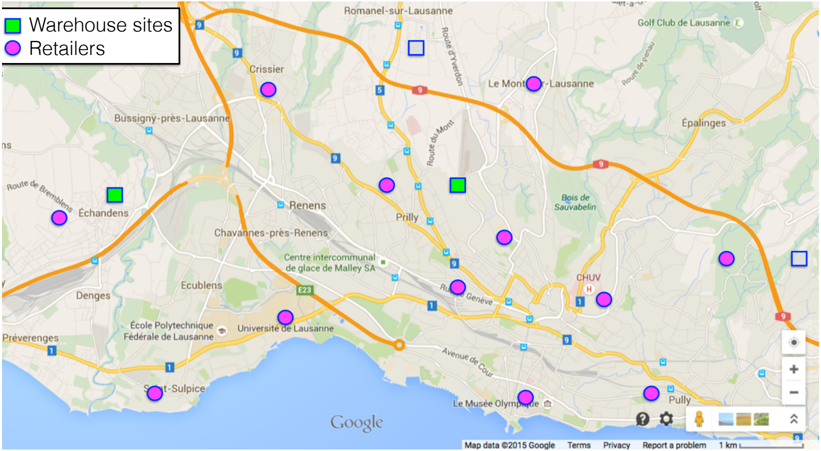
A Classical Distribution Problem

- A multinational retailing corporation wishes to construct new warehouses



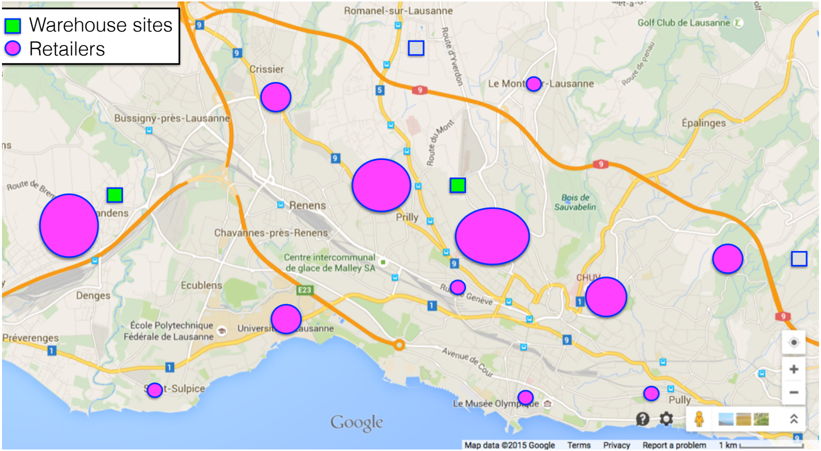
A Classical Distribution Problem

1. Choose where to build the new warehouses



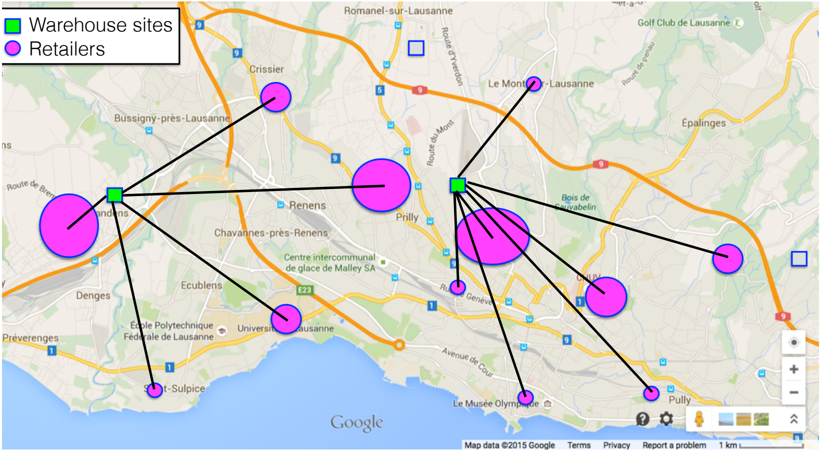
A Classical Distribution Problem

2. Observe amount of weekly demand



A Classical Distribution Problem

3. Transport goods to retailers to maximize profits



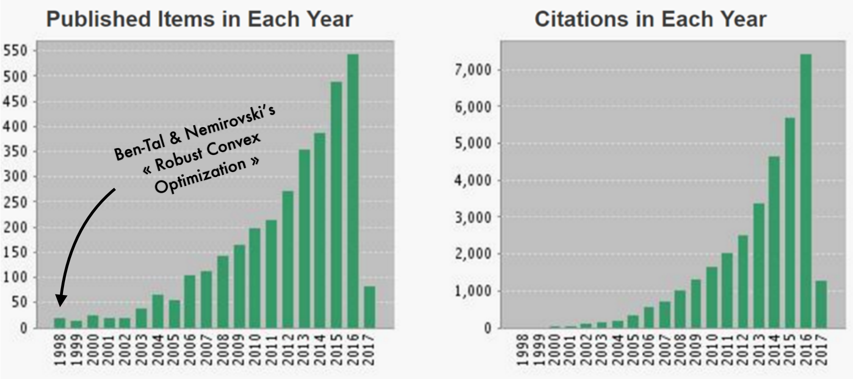
A Classical Distribution Problem

- Facility location-transportation model

$$
\begin{aligned}
\underset{I\in\{0,1\}^n, x, Y\geq 0}{\text{maximize}} \quad & \eta \overbrace{\sum_i \sum_j Y_{ij}}^{\text{sold product}} - \left(\overbrace{c^T x + K^T I}^{\text{location cost}} + \overbrace{\sum_i \sum_j (p_i + t_{ij}) Y_{ij}}^{\text{transportation\&production cost}} \right) \\
\text{s. t.} \quad & \sum_j Y_{ij} \leq x_i \ , \forall i, \qquad (\textit{Capacity constraint}) \\
& \sum_i Y_{ij} \leq d_j \ , \forall j, \quad (\textit{Demand constraint}) \\
& x_i \leq M I_i \ , \forall i, \qquad (\textit{Facility Size constraint})
\end{aligned}
$$

How can one account for demand uncertainty?

Robust Optimization Is Now a Well Established Methodology



A Classical Robust Distribution Problem

- Robust Facility location-transportation model:

$$\begin{aligned} \underset{I\in\{0,1\}^n,x}{\text{maximize}} \quad & \min_{d\in\mathcal{D}} h(I,x,d) \\ \text{s. t.} \quad & x_i \le MI_i\ ,\ \forall\, i, \qquad (\textit{Facility Size} \ \ \textit{constraint}) \end{aligned}$$

where $h(I,x,d)$ is the optimal value of

$$\begin{aligned} \max_{Y\ge 0} \quad & \eta \overbrace{\sum_i \sum_j Y_{ij}}^{\textit{sold product}} - \left(\overbrace{c^T x + K^T I}^{\textit{location cost}} + \overbrace{\sum_i \sum_j (p_i + t_{ij}) Y_{ij}}^{\textit{transportation\&production cost}} \right) \\ \text{s. t.} \quad & \sum_j Y_{ij} \le x_i\ ,\ \forall\, i, \qquad (\textit{Capacity} \ \ \textit{constraint}) \\ & \sum_i Y_{ij} \le d_j\,,\ \forall\, j, \quad (\textit{Demand} \ \ \textit{constraint}) \end{aligned}$$

Outline

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Introduction

Background on Two-Stage Robust Linear Programs

Copositive Programming Reformulations

Applications

Conclusion

Static Robust Linear Program

[Ben-Tal & Nemirovski (2000), 1296 citations !]

- Consider the following static problem:

$$\underset{x\in\mathcal{X},y}{\text{maximize}} \quad c^T x + f^T y \tag{1a}$$

$$\text{s. t.} \quad Ax + By \leq D(x)z\ ,\ \forall z \in \mathcal{Z} \tag{1b}$$

  where we assume $n_x + n_y$ decision variables, J constraints, and m uncertain parameters.

- If $\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, z \geq 0,\ Pz = q\}$ is a non-empty polyhedral set defined by K constraints, then

$$\text{Problem (1)} \equiv \underset{x\in\mathcal{X},y,\Lambda}{\text{maximize}} \quad c^T x + f^T y$$
$$\text{s. t.} \quad Ax + By + \Lambda q \leq 0$$
$$D(x) + \Lambda P \geq 0\,,$$

  where $\Lambda \in \mathbb{R}^{J\times K}$.

TWO-STAGE ROBUST LINEAR PROGRAMS

[BEN-TAL ET AL. (2004), 824 CITATIONS !]

- Consider the following two-stage problem:

$$
\begin{array}{llll}
(TSRLP) & \underset{x\in\mathcal{X},y(\cdot)}{\text{maximize}} & \min_{z\in\mathcal{Z}} c^T x + f^T y(z) \\
& \text{s. t.} & Ax + By(z) \leq D(x)z \ \forall z \in \mathcal{Z}
\end{array}
$$

  where $y : \mathbb{R}^m \to \mathbb{R}^{n_y}$

- This problem can also be represented as

$$
(TSRLP) \qquad \underset{x\in\mathcal{X}}{\text{maximize}} \quad \min_{z\in\mathcal{Z}} h(x,z)
$$

  where

$$
\begin{array}{lll}
h(x,z) := \max_{y} & c^T x + f^T y \\
\text{s. t.} & Ax + By \leq D(x)z \, .
\end{array}
$$

Complexity of Two-Stage Robust Linear Programs

- Unfortunately, the two-stage robust linear program is known to be intractable in general [Ben-Tal et al. (2004)].
- Conservative approximation obtained by using affine adjustment functions :

$$y(z) := y + Yz$$

The two-stage robust problem reduces to

$$(AARC) \quad \underset{x\in\mathcal{X},y,Y}{\text{maximize}} \quad \min_{z\in\mathcal{Z}} c^T x + f^T(y + Yz)$$
$$\text{s. t.} \quad Ax + B(y + Yz) \le D(x)z \; \forall z \in \mathcal{Z}$$

- Some exact methods have been proposed but without polynomial time convergence guarantees [Zeng & Zhao (2013)]

OUTLINE

INTRODUCTION

BACKGROUND ON TWO-STAGE ROBUST LINEAR PROGRAMS

COPOSITIVE PROGRAMMING REFORMULATIONS

APPLICATIONS

CONCLUSION

Copositive Programming Reformulation I

- Assumptions
  1. $\mathcal{Z}$ is a non-empty and bounded polyhedral set
  2. The TSRLP problem is bounded above, i.e.
     $\forall x \in \mathcal{X}, \exists z \in \mathcal{Z}, h(x,z) < \infty$.
- Let our robust optimization problem take the form

$$\underset{x \in \mathcal{X}}{\text{maximize}} \qquad \psi(x) \ ,$$

where

$$\psi(x) \ := \ \min_{z \in \mathcal{Z}} \quad \max_{y} \qquad c^T x + f^T y \tag{2a}$$

$$\text{s. t.} \qquad Ax + By \leq D(x)z \tag{2b}$$

- Since (2) is bounded, strong LP duality applies

$$\psi(x) \ = \ \min_{z \in \mathcal{Z}, \lambda \geq 0} \qquad c^T x + z^T D(x)^T \lambda - (Ax)^T \lambda$$

$$B^T \lambda = f$$

Copositive Programming Reformulation I

- The function $\psi(x)$ minimizes a non-convex quadratic function over a polyhedron in the non-negative orthant

$$\begin{aligned} \psi(x) \;=\; \min_{\tilde{z}\geq 0} \quad & c^T x + \tilde{z}^T \tilde{Q}(x)\tilde{z} - \tilde{c}(x)^T \tilde{z} \\ & \tilde{A}\tilde{z} = \tilde{b}\,, \end{aligned}$$

where $\tilde{z} := [\lambda^T \quad z^T] \in \mathbb{R}^{J+m}$ and where

$$\tilde{Q}(x) := \begin{bmatrix} 0 & (1/2)D(x) \\ (1/2)D(x)^T & 0 \end{bmatrix} \quad \tilde{c}(x) := \begin{bmatrix} -(1/2)Ax \\ 0 \end{bmatrix}$$

$$\tilde{A} := \begin{bmatrix} B^T & 0 \\ 0 & P \end{bmatrix} \qquad \tilde{b} := \begin{bmatrix} d \\ q \end{bmatrix}$$

COPOSITIVE PROGRAMMING REFORMULATION I

- The function $\psi(x)$ minimizes a non-convex quadratic function over a polyhedron in the non-negative orthant

$$\begin{aligned} \psi(x) \;=\; \min_{\tilde{z}\geq 0} \quad & c^T x + trace(\tilde{Q}(x)^T \tilde{z}\tilde{z}^T) - \tilde{c}(x)^T \tilde{z} \\ & \tilde{A}\tilde{z} = \tilde{b} \\ & \tilde{A}\tilde{z}\tilde{z}^T = \tilde{b}\tilde{z}^T , \end{aligned}$$

where $\tilde{z} := [\lambda^T \quad z^T] \in \mathbb{R}^{J+m}$ and where

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$$\tilde{A} := \begin{bmatrix} B^T & 0 \\ 0 & P \end{bmatrix} \qquad \tilde{b} := \begin{bmatrix} d \\ q \end{bmatrix}$$

Copositive Programming Reformulation I

- The function $\psi(x)$ has an equivalent convex optimization reformulation ($\tilde{Z} := \tilde{z}\tilde{z}^T$) [Burer (2009)]

$$\begin{aligned}
\psi(x) \;=\; \min_{\tilde{Z},\tilde{z}} \quad & c^T x + trace(\tilde{Q}(x)^T \tilde{Z}) - \tilde{c}(x)^T \tilde{z} \\
& \tilde{A}\tilde{z} = \tilde{b} \\
& \tilde{A}\tilde{Z} = \tilde{b}\tilde{z}^T \\
& \begin{bmatrix} \tilde{Z} & \tilde{z} \\ \tilde{z}^T & 1 \end{bmatrix} \in \mathcal{K}_{\text{CP}} \;\;\&\;\; \text{rank}\left(\begin{bmatrix} \tilde{Z} & \tilde{z} \\ \tilde{z}^T & 1 \end{bmatrix}\right) = 1
\end{aligned}$$

where $\mathcal{K}_{\text{CP}}$ is the cone of completely positive matrices, i.e.

$$\mathcal{K}_{\text{CP}} := \left\{ M \;\middle|\; M = \sum_{k\in K} \tilde{z}_k \tilde{z}_k^T \text{ for some } \{\tilde{z}_k\}_{k\in K} \subset \mathbb{R}_+^{J+m+1} \right\}.$$

Copositive Programming Reformulation I

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$$\begin{aligned}
\psi(x) \; = \; \min_{\tilde{Z},\tilde{z}} \quad & c^T x + trace(\tilde{Q}(x)^T \tilde{Z}) - \tilde{c}(x)^T \tilde{z} \\
& \tilde{A}\tilde{z} = \tilde{b} \\
& \tilde{A}\tilde{Z} = \tilde{b}\tilde{z}^T \\
& \begin{bmatrix} \tilde{Z} & \tilde{z} \\ \tilde{z}^T & 1 \end{bmatrix} \in \mathcal{K}_{\text{CP}}
\end{aligned}$$

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Copositive Programming Reformulation I

- By conic duality we get

$$
\begin{aligned}
\psi(x) \geq \max_{\tilde{W},\tilde{w},\tilde{v},t} \quad & \tilde{c}(x)^T x + \tilde{b}^T \tilde{w} - t \\
\text{s. t.} \quad & \tilde{v} = \tilde{c}(x) - (1/2)(\tilde{A}^T \tilde{w} - \tilde{W}^T \tilde{b}) \\
& \begin{bmatrix} \tilde{Q}(x) - (1/2)(\tilde{W}^T \tilde{A} + \tilde{A}^T \tilde{W}) & \tilde{v} \\ \tilde{v}^T & t \end{bmatrix} \in \mathcal{K}_{\text{Cop}} \, ,
\end{aligned}
$$

where $\mathcal{K}_{\text{Cop}}$ is the cone of copositive matrices, i.e.

$$
\mathcal{K}_{\text{Cop}} := \left\{ M \,\middle|\, M = M^T, \; z^T M z \geq 0 \, , \, \forall\, z \in \mathbb{R}_+^{J+m+1} \right\} \, .
$$

Copositive Programming Reformulation I

Theorem 1 *[Xu & Burer (2016), Hanasusanto & Kuhn (2016)]*
If the TSRLP problem has "complete recourse", i.e.

$$\exists\, y \in \mathbb{R}^{n_y}\,,\ By < 0\,,$$

then the copositive program

$$(Copos_1)\quad \underset{x\in\mathcal{X},\tilde{W},\tilde{w},\tilde{v},t}{\text{maximize}} \quad c^T x + \tilde{b}^T\tilde{w} - t$$

$$\text{s. t.} \quad \tilde{v} = \tilde{c}(x) - (1/2)(\tilde{A}^T\tilde{w} - \tilde{W}^T\tilde{b})$$

$$\begin{bmatrix} \tilde{Q}(x) - (1/2)(\tilde{W}^T\tilde{A} + \tilde{A}^T\tilde{W}) & \tilde{v} \\ \tilde{v}^T & t \end{bmatrix} \in \mathcal{K}_{Cop}\,,$$

provides an exact reformulation of the TSRLP problem. Otherwise, $Copos_1$ only provides a conservative approximation.

Relation to AARC

Theorem 2 *[Xu & Burer (2016)]*
When $\mathcal{K}_{Cop}$ is replaced with $\mathcal{N} := \mathbb{R}_+^{J+m+1 \times J+m+1} \subset \mathcal{K}_{Cop}$ the copositive programming reformulation is equivalent to AARC.

- Hence, for any cone $\mathcal{K}$ such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{K}_{\text{Cop}}$, *Copos*$_1$ with $\mathcal{K}$ provides a tighter approximation than AARC
- There exists a hierarchy of semidefinite and polyhedral cones $\{\mathcal{K}_i\}_{i=1}^{\infty}$, with $\mathcal{N} \subseteq \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{\text{Cop}}$, such that for all $M \in \mathcal{K}_{\text{Cop}}$, there is a i^*, $M \in \mathcal{K}_{i^*}$
[Parrilo (2000), Bomze & de Klerk (2002)]
- This is valuable for complete recourse problems but what about relatively complete recourse problems ?

How to Fix Relatively Complete Recourse

- Assumption : The TSRLP problem has relatively complete recourse, i.e.

$$\forall x \in \mathcal{X}\, \forall z \in \mathcal{Z},\ \exists y,\ Ax + By \leq D(x)z$$

- This ensures that :

$$\begin{aligned} h(x,z) \;=\; \min_{\lambda} \quad & c^T x + z^T D(x)^T \lambda - (Ax)^T \lambda \qquad \in \mathbb{R} \\ \text{s. t.} \quad & \lambda \in \mathcal{P} := \{\lambda \,|\, \lambda \geq 0,\ B^T \lambda = f\} \end{aligned}$$

- Hence, always an optimal solution $\lambda^*(x,z)$ at a vertex of $\mathcal{P}$
- Since number of vertices is finite, there exists $u \in \mathbb{R}^J_+$:

$$\begin{aligned} \psi(x) \;=\; \min_{z \in \mathcal{Z}} h(x,z) \;=\; \min_{z \in \mathcal{Z}, \lambda \in \mathcal{P}} \quad & c^T x + z^T D(x)^T \lambda - (Ax)^T \lambda \\ \text{s. t.} \quad & \lambda \leq u \end{aligned}$$

Copositive Programming Reformulation II

- The function $\psi(x)$ minimizes a non-convex quadratic function over a polyhedron in the non-negative orthant

$$\begin{aligned} \psi(x) \;=\; \min_{\bar{y} \geq 0} \quad & c^T x + \bar{y}^T \bar{Q}(x) \bar{y} - \bar{c}(x)^T \bar{y} \\ & \bar{A}\bar{y} = \bar{b}\,, \end{aligned}$$

where $\bar{y} := [\lambda^T \quad z^T \quad s^T] \in \mathbb{R}^{2J+m}$ and where

$$\bar{Q}(x) := \begin{bmatrix} 0 & (1/2)D(x) & 0 \\ (1/2)D(x)^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{c}(x) := \begin{bmatrix} -(1/2)Ax \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{A} := \begin{bmatrix} B^T & 0 & 0 \\ 0 & P & 0 \\ I & 0 & I \end{bmatrix} \qquad \bar{b} := \begin{bmatrix} d \\ q \\ u \end{bmatrix}.$$

Copositive Programming Reformulation II

Theorem 3 *[AJ&D (2016b)]*
If the TSRLP problem has relatively complete recourse, then the copositive program

$$
\begin{array}{lll}
(Copos_2) & \displaystyle\max_{x\in\mathcal{X},\bar{W},\bar{w},\bar{v},t} & c^T x + \bar{b}^T \bar{w} - t \\
& \text{s. t.} & \bar{v} = \bar{c}(x) - (1/2)(\bar{A}^T \bar{w} - \bar{W}^T \bar{b}) \\
& & \begin{bmatrix} \bar{Q}(x) - (1/2)(\bar{W}^T \bar{A} + \bar{A}^T \bar{W}) & \bar{v} \\ \bar{v}^T & t \end{bmatrix} \in \mathcal{K}_{Cop} \,,
\end{array}
$$

provides an exact reformulation of the TSRLP problem.

The Penalized AARC model

Theorem 4 *[AJ&D (2016b)]*
When $\mathcal{K}_{Cop}$ is replaced with $\mathcal{N}$ the Copos$_2$ reformulation is equivalent to applying affine adjustments to:

$$(TSRLP') \qquad \underset{x\in\mathcal{X},y(\cdot),\theta(\cdot)}{\text{maximize}} \quad \min_{z\in\mathcal{Z}} c^T x + f^T y(z) - u^T\theta(z)$$
$$\text{s. t.} \quad Ax + By(z) \leq D(x)z + \theta(z) \; \forall z \in \mathcal{Z} \,.$$

Moreover, affine (and static) adjustments are always feasible in TSRLP′.

- u can be interpreted as a marginal penalty for violating constraints
- TSRLP′ $\equiv$ TSRLP since u is such that there always exists an optimal solution triplet with $\theta(z) := 0$.
- Method for converting a relatively complete recourse multi-stage linear program into a complete recourse one

OUTLINE

Robust Facility Location-Transportation Problem

- In AJ&D (2016b), we identify an instance for which

	AARC model	Penalized AARC (a.k.a. $Copos_2(\mathcal{N})$)	Exact model
Bound on w.-c. profit	0	6600	6600
W.-c. profit of x^*	0	6600	6600

- We recently randomly generated 10 000 problem instances, 5 facilities & 10 customer locations.

Optimality gap	Proportion of instances	
	AARC	Penalized AARC
$= 0\%$	20.6%	23.8%
$\leq 0.1\%$	20.9%	27.4%
$\leq 1\%$	28.4%	56.3%
Avg. Gap	10.5%	1.6%
Max Gap	50.0%	13.3%

WHAT SIZE PROBLEMS CAN WE SOLVE ? [AJ&D (2017)]

(T,L,N)	Γ	Penalized AARC		Exact
		Full form	Row generation	C&CG
(1,50,100)	10	-	3 241 sec	8 465 sec
	30	-	4 563 sec	-
	50	-	8 460 sec	-
	70	-	3 781 sec	7 682 sec
	90	-	1 382 sec	7 sec
	100	-	$<$ 1 sec	2 sec
	Avg.	-	3 572 sec	-
(20,15,30)	60	-	3 781 sec	184 sec
	180	-	5 646 sec	-
	300	-	10 567 sec	-
	420	-	4 445 sec	-
	540	-	663 sec	-
	600	-	1 sec	$<$1 sec
	Avg.	-	4 184 sec	-

(– stands for more than two days of computation)

Robust Multi-item Newsvendor

- In AJ&D (2016a): the robust multi-item newsvendor problem with uncorrelated demand can be solved optimally by AARC/*Copos*($\mathcal{N}$) when using budgeted uncertainty set with integer Γ.

- In AJ&D (2016b): if demand is correlated than solution improves using *Copos* with $\mathcal{K} \supset \mathcal{N}$:

	AARC	$Copos(\mathcal{K}^4_{\text{LP}})$	$Copos(\mathcal{K}^1_{\text{SDP}})$	Exact
W.-c. profit bound	41.83	41.83	411.08	825.83
Actual w.-c. profit	41.83	41.83	664.76	825.83

Outline

CONCLUSION & OPEN QUESTIONS

1. Copositive programming is a useful tool for generating conservative approximations for TSRLP
   - *Copos*($\mathcal{K}$) with $\mathcal{K} \supset \mathcal{N}$ always improves on AARC
   - Although hierarchy of polyhedral cones $\mathcal{N} \subset \mathcal{K}^d_{\mathrm{LP}} \subset \mathcal{K}_{\mathrm{Cop}}$ provide LP reformulations, preliminary results indicate that classical ones perform poorly
   - Can *Copos*($\mathcal{K}$) provide intuition on approximate policies ?
   - Do *Copos*($\mathcal{K}$) reformulations exist for multi-stage problems?
2. Penalized violations transform any two-stage LP with relatively complete recourse in one with complete recourse
   - A useful preprocessing step for AARC when feasibility is a challenge
   - Is it possible to generalize this approach to robust multi-stage non-linear problems?

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Questions & Comments ...

... Thank you!