“Dice”-sion Making under Uncertainty: When Can a Random Decision Reduce Risk?

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CRC in decision making under uncertainty

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Facility Location under Uncertainty

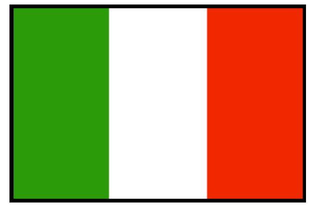
Where should facility be built so as to maximize our expected profits?

One can control risk exposure using
mean semi-variance

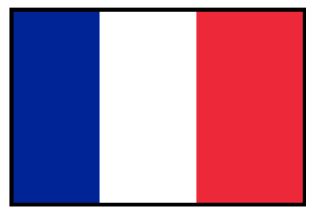
$$\mathbb{E}[X] - \mathbb{E}\left[\mathbb{E}[X] - X\right]_+^2$$

0.45 vs. 0.59
μ = 0.52

0.04 vs. 1.79
μ = 0.92



-0.05 vs. 3.50
μ = 1.73



-0.10 vs. 5.39
μ = 2.65

-0.51 vs. 6.62
μ = 3.06

Facility Location under Uncertainty

Where should it be built so as to maximize the mean semi-variance?

One can control risk exposure using mean semi-variance

$$\mathbb{E}[X] - \mathbb{E}\left[\mathbb{E}[X] - X\right]_+^2$$

0.45 vs. 0.59
M/SV = 0.52

0.04 vs. 1.79
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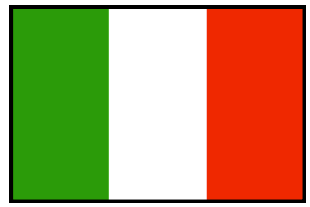
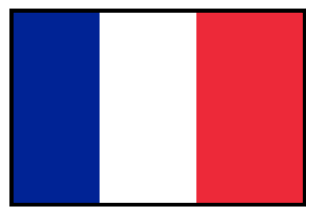


-0.05 vs. 3.50
M/SV = 0.15

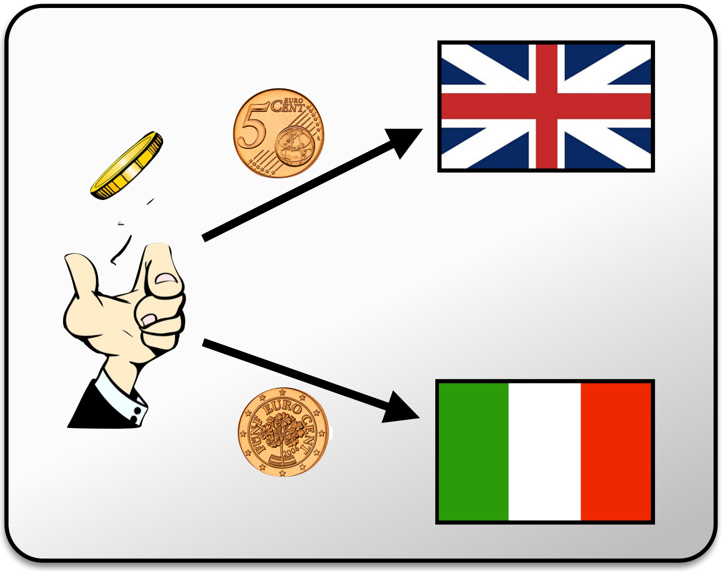


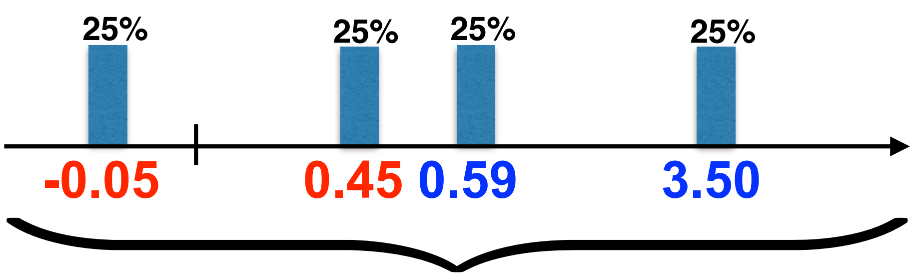
-0.10 vs. 5.39
M/SV = -1.13

-0.51 vs. 6.62
M/SV = -3.30

Facility Location under Uncertainty

Making a decision randomly can actually reduce the risk:





M/SV = 0.59

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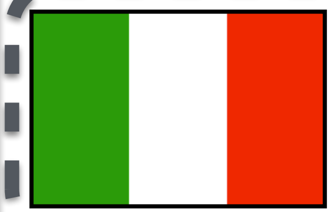

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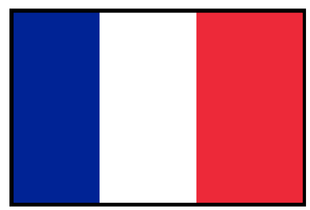

-0.10 vs. 5.39
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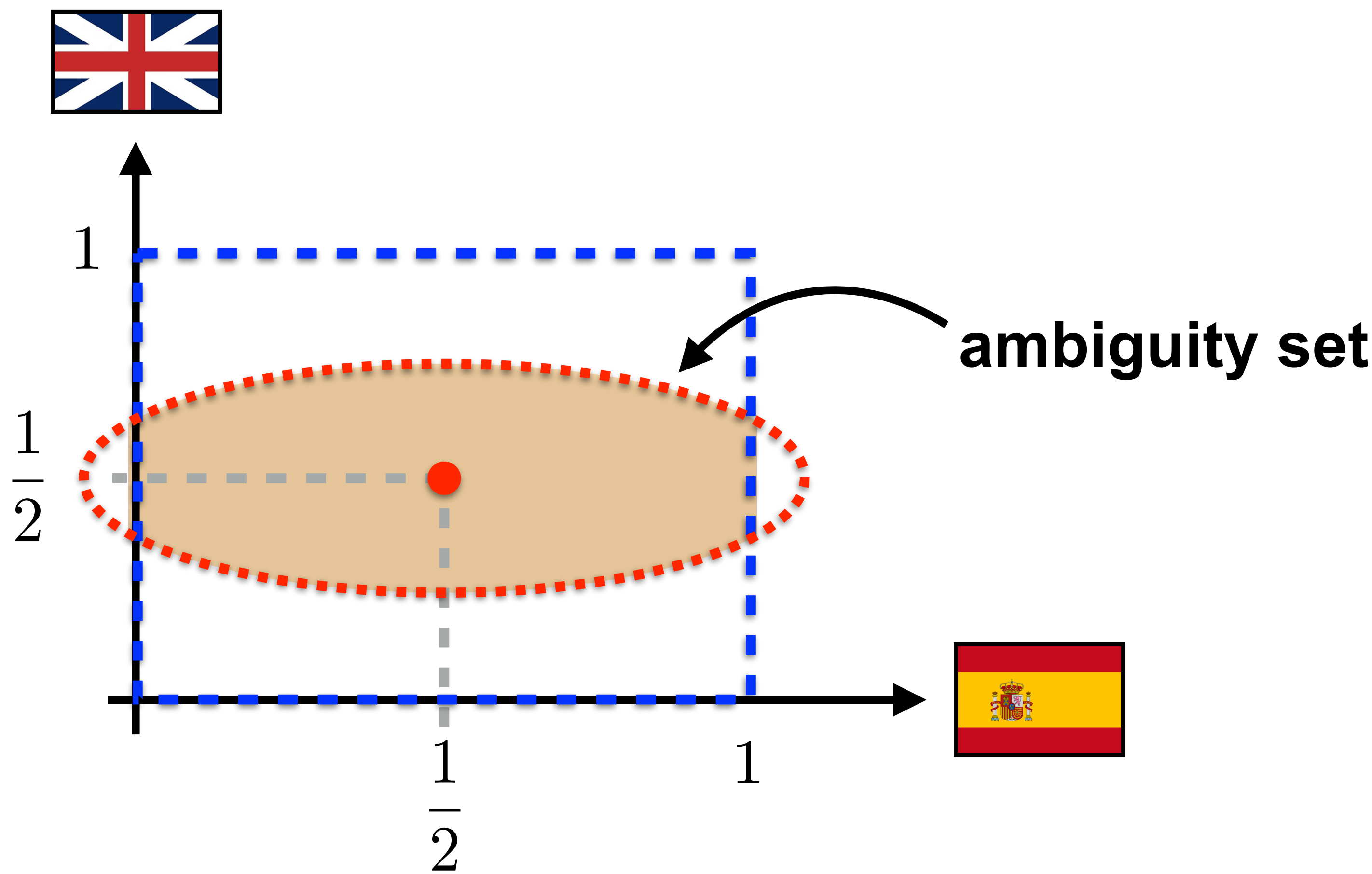
Ambiguity Averse Decision-Making

In practice, the probabilities for the profit scenarios may only be partially known:

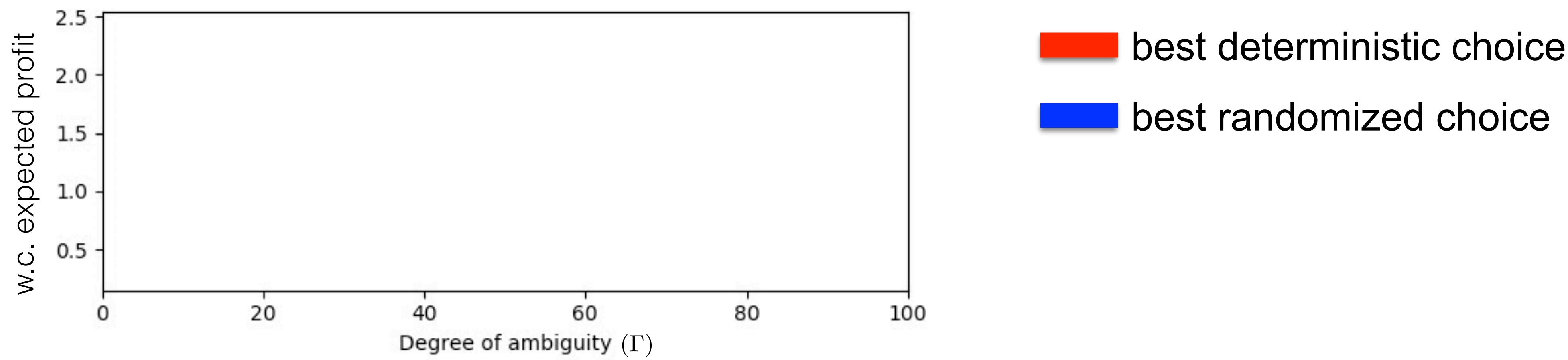


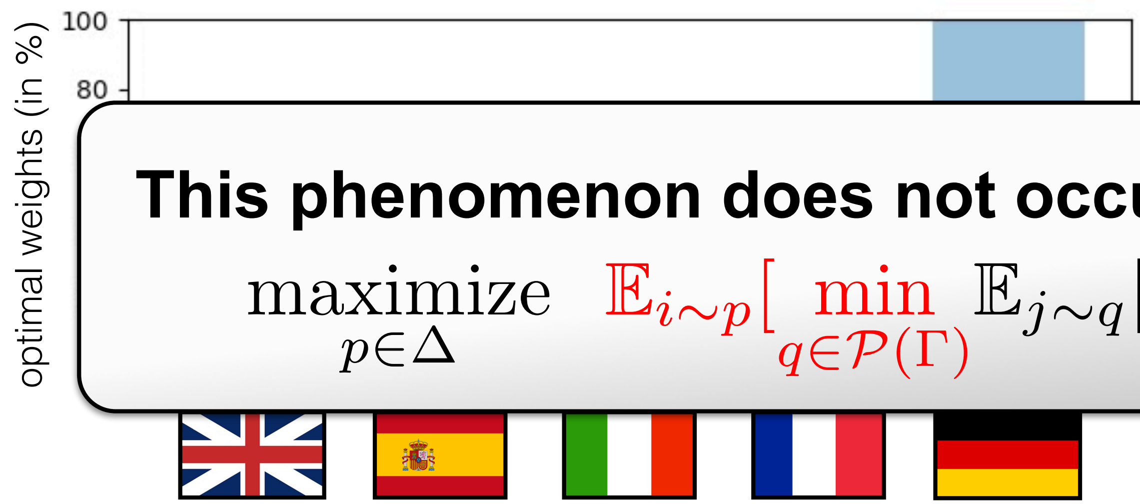
Distributionally robust optimization: Optimize a risk measure over worst distribution in ambiguity set

Ambiguity Averse Decision-Making

Assume we want to maximize expected profits under the worst probability distribution in the ambiguity set:

$$\underset{p\in\Delta}{\text{maximize}} \quad \min_{q\in\mathcal{P}(\Gamma)} \quad \mathbb{E}_{i\sim p,\, j\sim q}[\text{profit}(\text{loc}_i, \text{scen}_j)]$$





This phenomenon does not occur when one employs:

$$\underset{p\in\Delta}{\text{maximize}} \quad \mathbb{E}_{i\sim p}\Big[\min_{q\in\mathcal{P}(\Gamma)} \mathbb{E}_{j\sim q}[\text{profit}(\text{loc}_i, \text{scen}_j)]$$

Agenda

1. ~~Motivation~~

2. Randomization under Distributional Ambiguity

   - [x] Ambiguity Averse Risk Measures
   - [x] Problem Setup
   - [x] The Power of Randomization

3. Discussion

Agenda

1. ~~Motivation~~

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Ambiguous Probability Spaces

- **We model uncertainty via an *ambiguous* probability space:**

$$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$$

- **We denote by $\mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ the real-valued random variables that are essentially bounded w.r.t. *all* $\mathbb{P} \in \mathcal{P}_0$**

- **We denote by $F_X^{\mathbb{P}} \in \mathcal{D}$ the distribution function of X under $\mathbb{P}$:**

$$F_X^{\mathbb{P}}(x) = \mathbb{P}(X \leq x) \quad \forall x \in \mathbb{R}$$

- **An ambiguous probability space $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic if:**

$\exists U_0 \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ that follows a uniform distribution on $[0, 1]$ under *every* probability measure $\mathbb{P} \in \mathcal{P}_0$.

Risk Measures

A risk measure assigns each random variable a risk index:

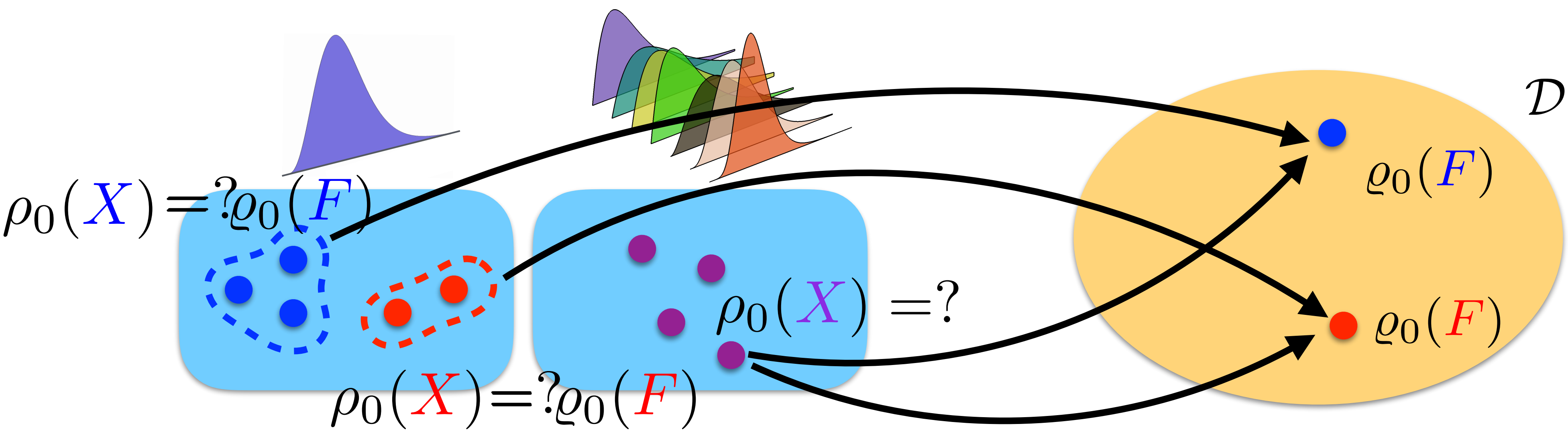
$$\rho_0 : \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) \to \mathbb{R}$$

A risk measure ρ_0 is law invariant if it satisfies:

$$\{F_X^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\} = \{F_Y^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\} \;\Rightarrow\; \rho_0(X) = \rho_0(Y)$$

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is law invariant. Then, there exists a unique $\varrho_0 : \mathcal{D} \to \mathbb{R}$ satisfying

$$\rho_0(X) = \varrho_0(F_X) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0) : F_X^{\mathbb{P}} = F_X \;\; \forall \mathbb{P} \in \mathcal{P}_0.$$



Ambiguity Averse Risk Measures

Definition: A risk measure ρ_0 is called ambiguity averse if it satisfies for all $X, Y \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$:

- **Ambiguity aversion:** If $\{F_X^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\} \subseteq \{F_Y^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}_0\}$, then $\rho_0(X) \leq \rho_0(Y)$.
- **Ambiguity monotonicity:** If $\varrho_0(F_X^{\mathbb{P}}) \leq \varrho_0(F_Y^{\mathbb{P}})$ for all $\mathbb{P} \in \mathcal{P}_0$, then $\rho_0(X) \leq \rho_0(Y)$.

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is law invariant, ambiguity averse and translation invariant.

Then the risk measure satisfies

$$\rho_0(X) = \sup_{\mathbb{P} \in \mathcal{P}_0} \varrho_0(F_X^{\mathbb{P}}) \quad \forall X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0).$$

Ambiguity Averse Risk Measures

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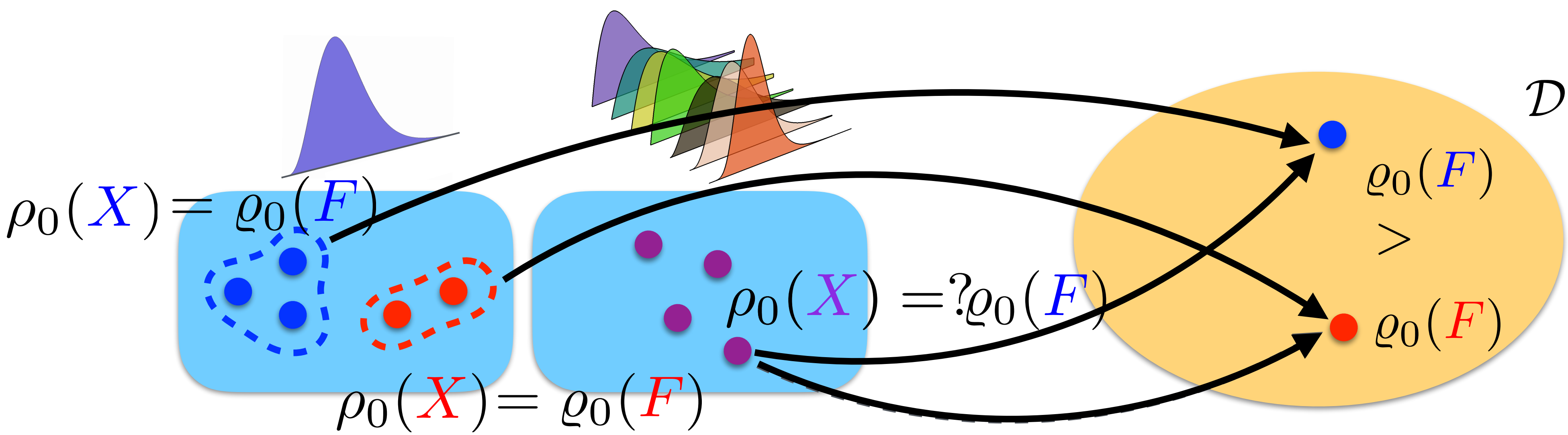

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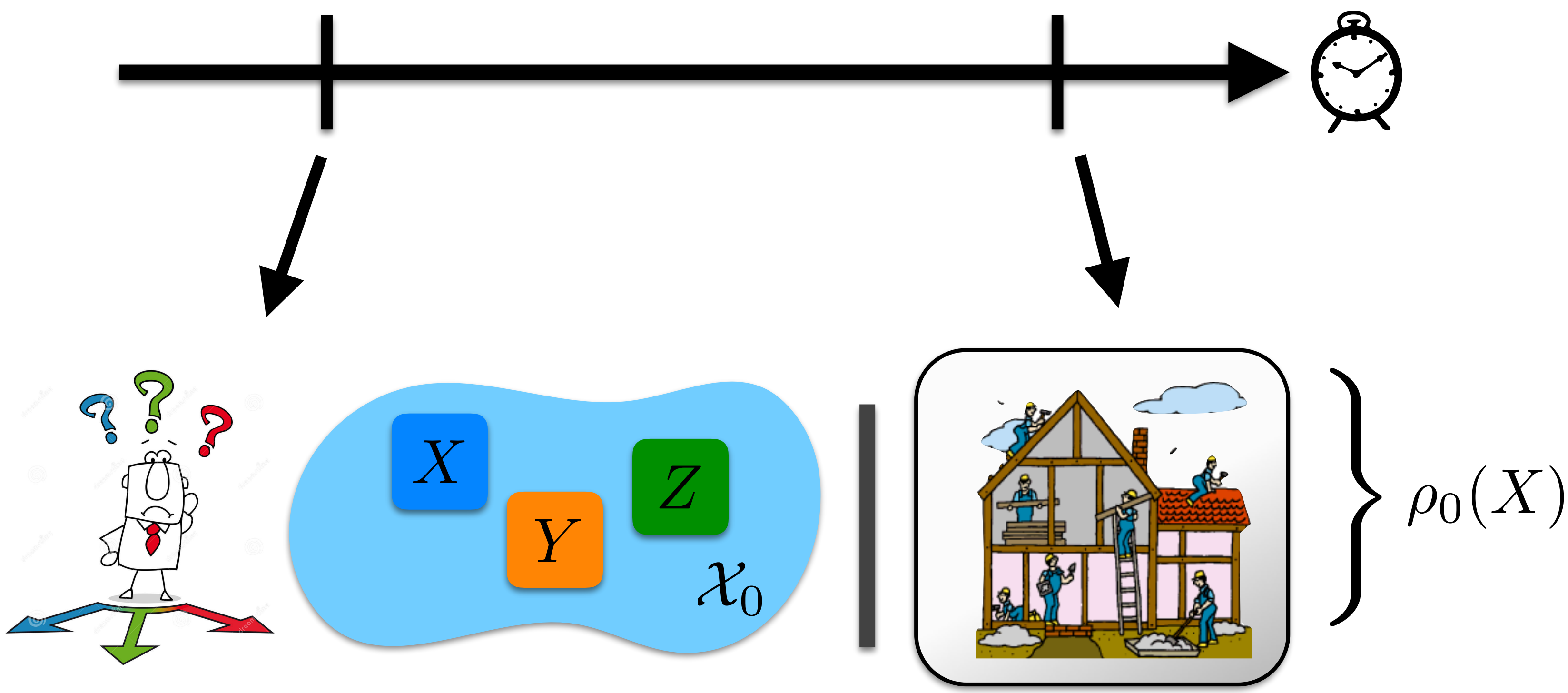
From Deterministic to Random Decisions

We consider an ambiguity averse risk minimization problem

$$\underset{X \in \mathcal{X}_0}{\text{minimize}} \;\; \rho_0(X) \qquad \text{(PSP)}$$

where $\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ **denotes the feasible region.**



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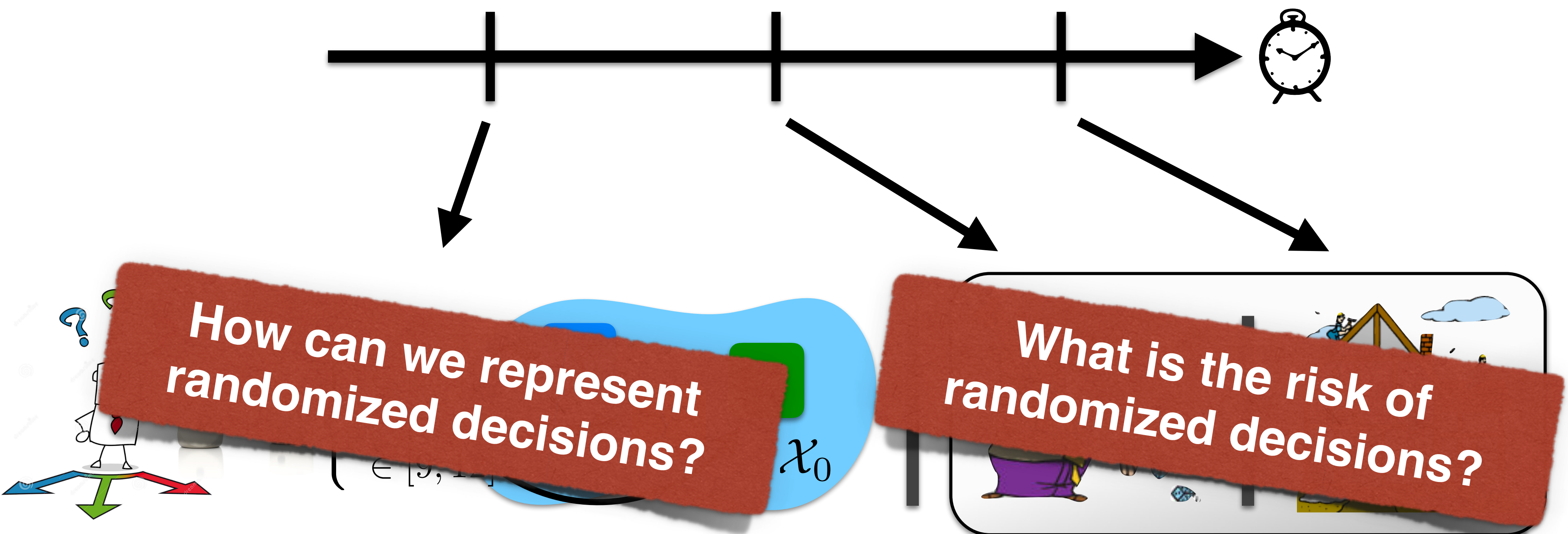
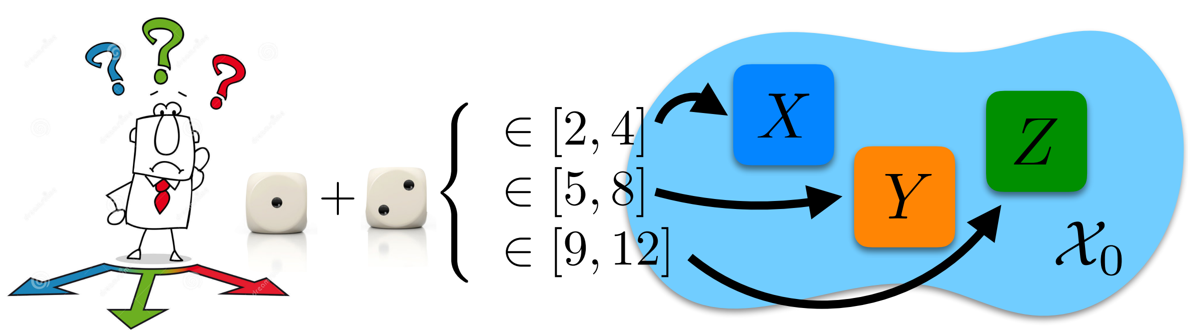


Randomization Devices

We assume we have a randomisation device that generates uniform samples from $[0,1]$:

pure strategy problem	**randomized strategy problem**
$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$	
$\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$	



Randomization Devices

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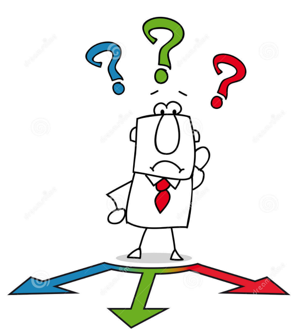
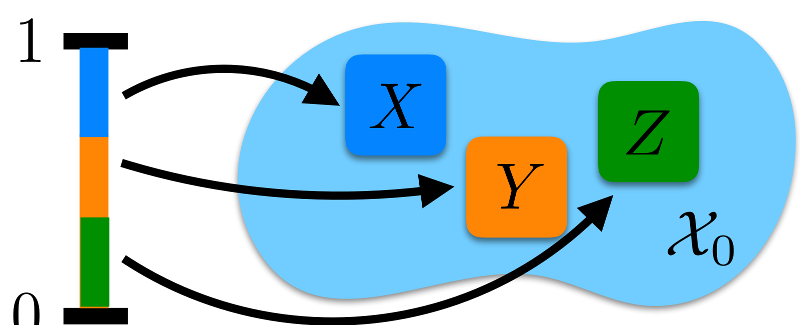
pure strategy problem	**randomized strategy problem**
$(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$	$(\Omega, \mathcal{F}, \mathcal{P})$ with ☑ $\Omega = \Omega_0 \times [0,1]$ ☑ $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{B}_{[0,1]}$ ☑ $\mathcal{P} = \{\mathbb{P} \times \mathbb{U} \,:\, \mathbb{P} \in \mathcal{P}_0\}$
$\mathcal{X}_0 \subseteq \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$	$\mathcal{X} = \{X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) \,:\, X(\cdot, u) \in \mathcal{X}_0 \;\; \forall u \in [0,1]\}$



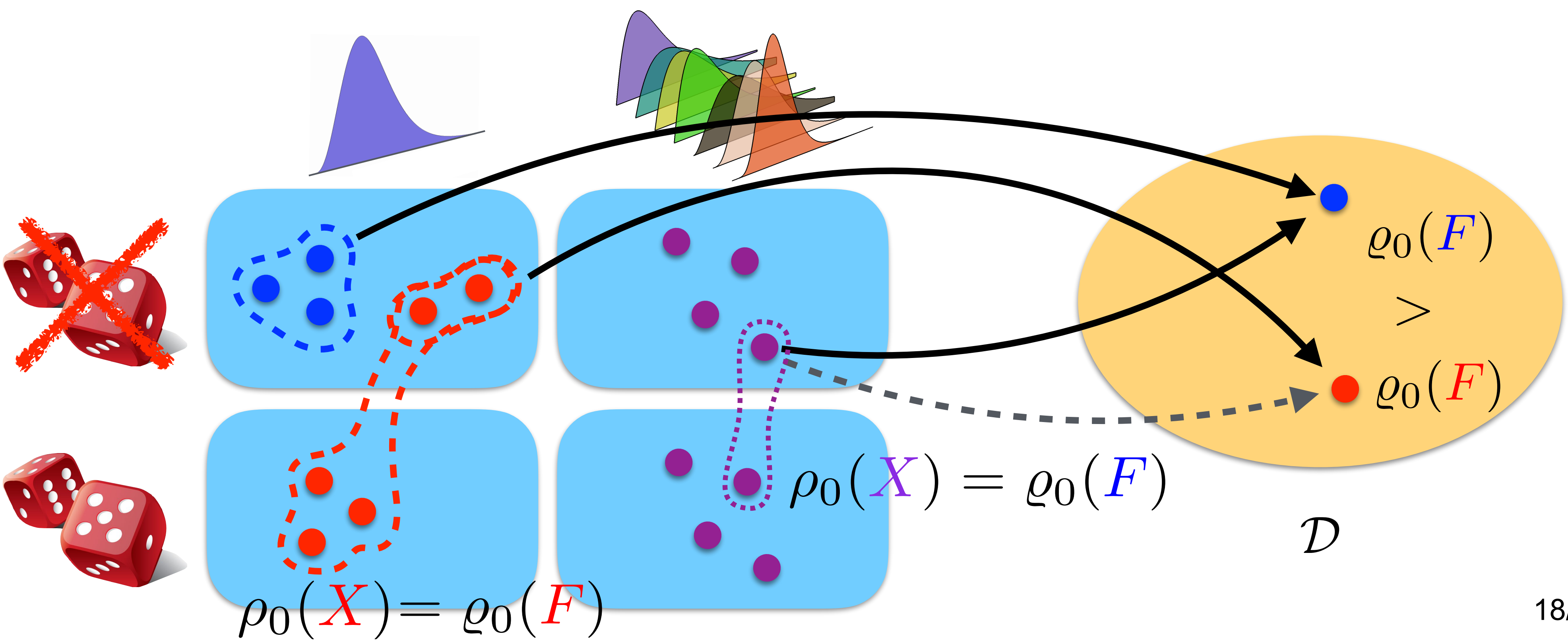


Risk of Randomized Decisions

Proposition: Assume that $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is non-atomic and that ρ_0 is law invariant, ambiguity averse and translation invariant.

The unique extension of ρ_0 to an ambiguity averse risk measure ρ on $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P})$ is given by

$$\rho(X) = \sup_{\mathbb{P}\in\mathcal{P}} \varrho_0(F_X^{\mathbb{P}}) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).$$



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   - [x] ~~Ambiguity Averse Risk Measures~~
   - [x] ~~Problem Setup~~
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3. Discussion

Randomized Strategy Problem

We define the randomized strategy problem

$$\underset{X \in \mathcal{X}}{\text{minimize}} \;\; \rho(X) \qquad \text{(RSP)}$$

where the extended risk measure ρ is defined via

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \varrho_0(F_X^{\mathbb{P}}) \quad \forall X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}).$$

and $\mathcal{X}$ denotes the enlarged feasible region:

$$\mathcal{X} = \left\{ X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathbb{P}) \, : \, X(\cdot, u) \in \mathcal{X}_0 \;\; \forall u \in [0, 1] \right\}$$

Theorem: If ρ_0 is convex and $\mathcal{X}_0$ is convex, then

(PSP) = (RSP).

The Power of Randomization

Theorem: Assume that

- ☑ $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ has a maximally ambiguous random variable:

  $\exists\, X \in \mathcal{L}_\infty(\Omega_0, \mathcal{F}_0, \mathcal{P}_0),\ \left\{F_X^{\mathbb{P}} \,:\, \mathbb{P} \in \mathcal{P}_0\right\} = \mathcal{D}$

- ☑ ρ_0 satisfies the Lebesgue property:

  $\lim_{k\to\infty} \varrho_0(F_k) = \varrho_0(F)$ whenever $F_k \to F$.

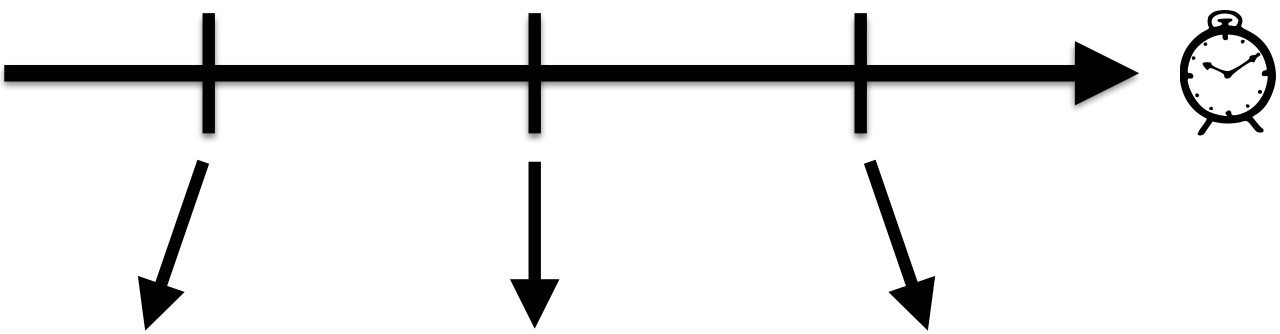
Then there is $\mathcal{X}_0$ such that (PSP) > (RSP).

The Rainbow Urn Game

Consider an urn with balls of K different colors where:

- the number of balls is unknown
- the proportions of colors are unknown

A player is offered the following game:



player names color

ball is drawn

player receives:

- -$1 if ball is of stated color
- +$1 if ball is *not* of stated color

The Rainbow Urn Game

Assume the player uses an ambiguity averse risk measure ρ_0:

Strategy → **Worst-case outcome**

Any *pure strategy* is as good as losing $1 with certainty → All balls are of stated color

The *randomized strategy* that names each color with probability $1/K$ →

- -$1 with probability $\frac{1}{K}$
- +$1 with probability $\frac{K-1}{K}$

If ρ_0 has the Lebesgue property, then this is as attractive as receiving +$1 for sure as $K \to \infty$!

Randomization is a cure for ambiguity !!!

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   - ☑ ~~The Power of Randomization~~
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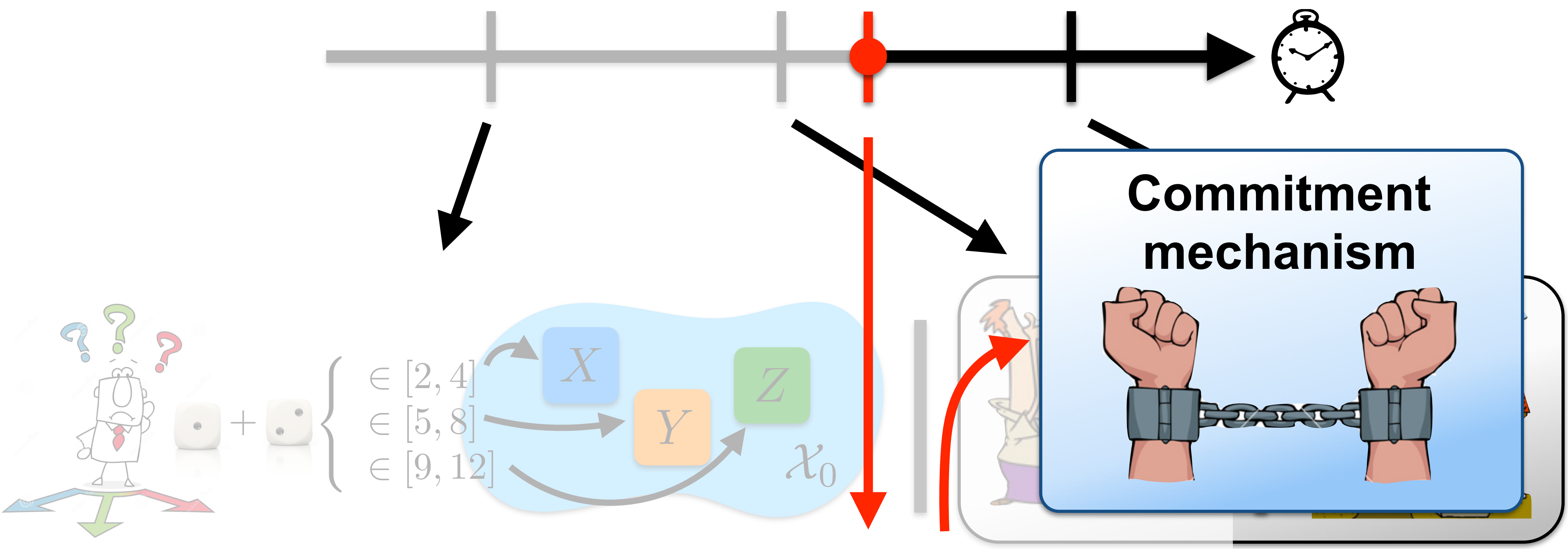
Summary

	Randomization Receptive	Randomization Proof
Distributional Ambiguity	every RM with Lebesque property	convex RM's with convex $\mathcal{X}_0$
Stochastic Uncertainty	Mean moment & mean deviation RM's	convex RM's with convex $\mathcal{X}_0$
	Mean semi-moment & mean semi-deviation RM's	Mixture quasi-concave RM's

The Issue of Time Consistency

Remember the randomized strategy problem:

$$\underset{X \in \mathcal{X}}{\text{minimize}} \quad \rho(X) \qquad \text{(RSP)}$$



Once we observe the outcome of the randomization, we have an incentive to deviate in favour of the optimal pure choice!

Bibliography

Randomized decisions in economics:

[1] M. Agranov and P. Ortoleva. *Stochastic choice and preferences for randomization.* Available on SSRN, 2015.

Randomized decisions in algorithm design:

[2] R. Motwani and P. Raghavan. *Randomized algorithms.* Cambridge University Press, 1995.

Randomized decisions in Markov decision processes:

[3] Y. Le Tallec. *Robust, risk-sensitive, and data-driven control of Markov decision processes.* PhD thesis, MIT, 2007.

[4] W. Wiesemann, D. Kuhn and B. Rustem. *Robust Markov decision processes.* Math. OR 62(6):1358-1376, 2014.

Randomized decisions in SP and DRO:

[7] G. Pflug. Version-independence and nested distributions in multistage stochastic optimization. SIAM J. Optim. 20(3):1406-1420, 2010.

[8] E. Delage, D. Kuhn and W. Wiesemann. *"Dice"-sion making under uncertainty: When can a random decision reduce risk?* Available on Opt. Online, 2016.

Accounting for time inconsistencies with a commitment mechanism:

[9] R. H. Strotz. *Myopia and Inconsistency in Dynamic Utility Maximization.* The Review of Economic Studies, 23(3):165-180, 1955.