Equal Risk Pricing and Hedging of Financial Derivatives with Convex Risk Measures

The case of worst-case risk measures

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Outline

WHAT IS AN OPTION?

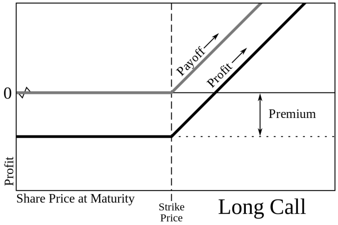

Figure: Profits from buying a call option

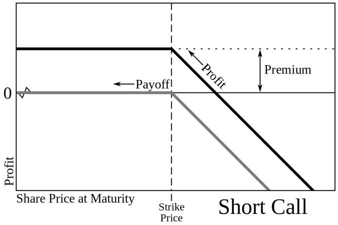

Figure: Profits from writing a call option

$$F(S_K) = \max\{0, S_K - StrikePrice\}$$

Graphs are from: https://en.wikipedia.org/wiki/Call_option

DIFFERENT MARKETS

- **Complete market**?
  - According to Rutkowski [2010] “A market model is said to be complete if any financial derivative admits a replicating strategy. The completeness of a market model ensures that any derivative security can be priced by arbitrage and hedged through dynamic trading in primary traded assets”.
  - Perfect replication
  - Unique price for derivative products (The option price is basically the initial wealth by which we can replicate the option payoff)
- **Incomplete market**?
  - Replication error
  - Different prices depending on the risk aversion of the investors

OPTION PRICING METHODS

- ϵ-arbitrage [Föllmer et al., 1985, Schweizer, 1996, Gourieroux et al., 1998, Bertsimas et al., 2001]
- Equal risk pricing (ERP) [Guo and Zhu, 2017]

ϵ-ARBITRAGE MODEL (WORST-CASE)

Robust ϵ-arbitrage is to replicate the option payoff F_K so that the portfolio value at the maturity, X_K, is as close as possible to the payoff according to a **symmetric measure**:

$$\epsilon(w_0) = \min_{X \in \mathcal{X}(w_0)} \sup_{S_{1:K} \in \mathcal{U}} [(X_K - F(S_K))^2] \quad (1)$$

The initial capital for constructing such a portfolio, $w_0^* = \text{argmin}_{w_0} \epsilon(w_0)$, is considered the **production cost** of the option [Bertsimas et al., 2001].
$\mathcal{X}(w_0)$ is the set of all admissible self financing hedging strategies starting with the initial capital w_0.

OUTLINE

The minimal worst-case risk

- The minimal worst-case risks achievable by the option writer and buyer are defined as:

$$\varrho^w(w_0) = \inf_{X \in \mathcal{X}(w_0)} \sup_{S_{1:K} \in \mathcal{U}^w} (F(S_K) - X_K) \tag{2a}$$

$$\varrho^b(w_0) = \inf_{X \in \mathcal{X}(-w_0)} \sup_{S_{1:K} \in \mathcal{U}^b} (-F(S_K) - X_K)\,, \tag{2b}$$

where $w_0 \in \mathbb{R}$ is the price charged to the buyer for an option with payout $F(S_K)$ at time K.

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  where $w_0 \in \mathbb{R}$ is the price charged to the buyer for an option with payout $F(S_K)$ at time K.
- When discrete trading is used,

$$\mathcal{X}(w_0) = \left\{ X : \Omega \to \mathbb{R}^K \,\middle|\, \exists \{\xi_k\}_{k=0}^{K-1}, X_k = w_0 + \sum_{k'=0}^{k-1} \xi_{k'} \Delta S_{k'+1}, \forall k = 1, \ldots, K \right\}$$

  where ξ_k and ΔS_{k+1} captures the number of risky assets held and its unknown price change in period $[t_k,\ t_{k+1}]$.

The uncertainty sets

The uncertainty set proposed in [Bandi and Bertsimas, 2014] is motivated by the central limit theorem:

$$\mathcal{U}_1 = \left\{ S \in \mathbb{R}^K \middle| \exists r, S_k = S_0 \prod_{\ell=1}^{k} (1 + r_\ell), \left| \frac{\sum_{\ell=1}^{k} \log(1 + r_\ell) - \mu k T/K}{\sigma \sqrt{kT/K}} \right| \leq \Gamma, \forall k \in \{1, ..., K\} \right\}$$

where Γ denotes a budget of the uncertainty.

The uncertainty sets

We propose an uncertainty set motivated by [Bernhard, 2003]:

$$\mathcal{U}_2 = \left\{ S \in \mathbb{R}^K \middle| \exists r, S_k = S_0 \prod_{\ell=1}^{k}(1 + r_\ell), \left| \frac{\sum_{\ell=1}^{hN} r_\ell^2 - \sigma^2 hT/S}{\sqrt{hN}} \right| \leq \Gamma, \forall h \in \{1, ..., H\} \right\}$$

where the time horizon is partitioned into $H \approx \sqrt{K}$ intervals.

THE UNCERTAINTY SETS

- For both uncertainty sets, one can define the Bellman equations associated to the writer and buyer's worst-case risk minimization problems using the projection methods described in Delage and Iancu [2015].
- The state space of the DP for $\mathcal{U}_1$: $[\sum_{\ell=1}^{k} \log(1 + r_\ell)]$
- The state space of the DP for $\mathcal{U}_2$: $[S_k, \sum_{\ell=1}^{k} r_\ell^2]$

THE EQUAL RISK PRICE

Definition (Equal risk price)

The equal risk price is defined as the unique w_0^* that satisfies

$$\varrho^w(w_0^*) = \varrho^b(w_0^*) \in \mathbb{R}\,, \tag{3}$$

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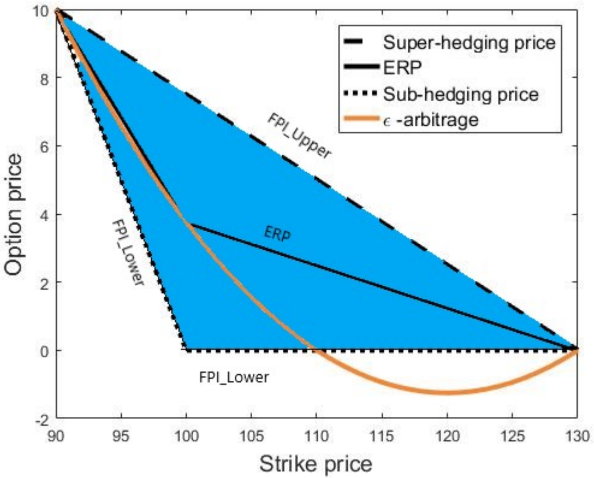
Lemma

If the fair price interval exists and is non-empty, then the equal risk price lies in the no-arbitrage price interval.

ERP ILLUSTRATION



OUTLINE

Comparing the different prices

Option prices for $S_0 = 1000$, $\mu = 0.07$, $\sigma = 0.13$, strike prices of 950, 1000, 1050.

With $\mathcal{U}_1$:

	ITM		**ATM**		**OTM**	
Periods	49	100	49	100	49	100
Γ	2.79	2.87	2.79	2.87	2.79	2.87
FPI-Upper	155.88	173.9	100.81	110.9	86.69	95.45
ERP	77.94	86.95	50.41	55.45	43.34	47.73
FPI-Lower	0	0	0	0	0	0
ϵ-arbitrage price	75.2	91.2	56	62.4	27.2	38.4
BS	78.8	78.8	51.15	51.15	31.17	31.17

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	ITM		**ATM**		**OTM**	
Periods	100	225	100	225	100	225
Γ	0.006	0.004	0.006	0.004	0.006	0.004
FPI-Upper	107.04	102.18	81.66	75.96	61.57	55.77
ERP	83.36	83.24	55.38	54.99	35.25	34.88
FPI-Lower	59.67	64.3	29.11	34.03	8.93	13.98
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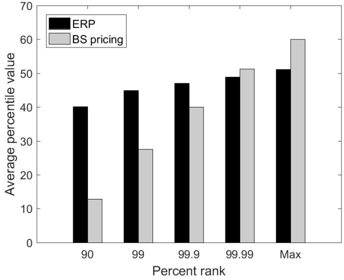
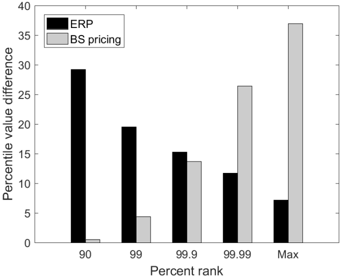
AVERAGE AND DIFFERENCES IN RISK EXPOSURE I

Comparison of hedging performance achieved under the Black-Scholes and the equal risk prices, using $\mathcal{U}_2$, of a European call option with $K = 16$ rebalancing periods.

At the money option:





OUTLINE

INTRODUCTION

THE WORST-CASE EQUAL RISK PRICE

NUMERICAL EXPERIMENTS

CONCLUSION

Conclusion

- We explored the pricing and hedging of options in an incomplete market on the case of equal risk pricing under worst-case risk measures
- We showed that ERP reduces to the center of the fair price interval when the worst-case is used as the risk measure, which can be obtained by solving two dynamic hedging problems
- We numerically showed that the equal risk model under the worst-case risk measure imposes more equal and on average lower risks to the option writer and the buyer.

BIBLIOGRAPHY

Chaithanya Bandi and Dimitris Bertsimas. Robust option pricing. *European Journal of Operational Research*, 239(3): 842–853, 2014.

Pierre Bernhard. A robust control approach to option pricing. *Applications of Robust Decision Theory and Ambiguity in Finance. City University Press, London*, 2003.

Dimitris Bertsimas, Leonid Kogan, and Andrew W Lo. Hedging derivative securities and incomplete markets: an ϵ-arbitrage approach. *Operations research*, 49(3):372–397, 2001.

Erick Delage and Dan A Iancu. Robust multistage decision making. In *The operations research revolution*, pages 20–46. INFORMS, 2015.

Hans Föllmer, Hans Sondermann, and Dieter Sondermann. *Hedging of non-redundant contingent claims*, pages 205 – 223. 06 1985. doi: 10.13140/RG.2.1.3298.8322.

Christian Gourieroux, Jean Paul Laurent, and Huyên Pham. Mean-variance hedging and numéraire. *Mathematical finance*, 8(3):179–200, 1998.

Ivan Guo and Song-Ping Zhu. Equal risk pricing under convex trading constraints. *Journal of Economic Dynamics and Control*, 76:136–151, 2017.

Marek Rutkowski. *Complete Markets*. American Cancer Society, 2010. ISBN 9780470061602. doi: 10.1002/9780470061602.eqf04006. URL `https://onlinelibrary.wiley.com/doi/abs/10.1002/9780470061602.eqf04006`.

Martin Schweizer. Approximation pricing and the variance-optimal martingale measure. *The Annals of Probability*, 24(1):206–236, 1996.

Link to the paper