Preference robust optimization for decision making under uncertainty

Erick Delage*
(joint work with Benjamin Armbruster** and Jonathan Y. Li***)

* HEC Montréal, CRC in decision making under uncertainty
** Northwestern University   *** University of Ottawa

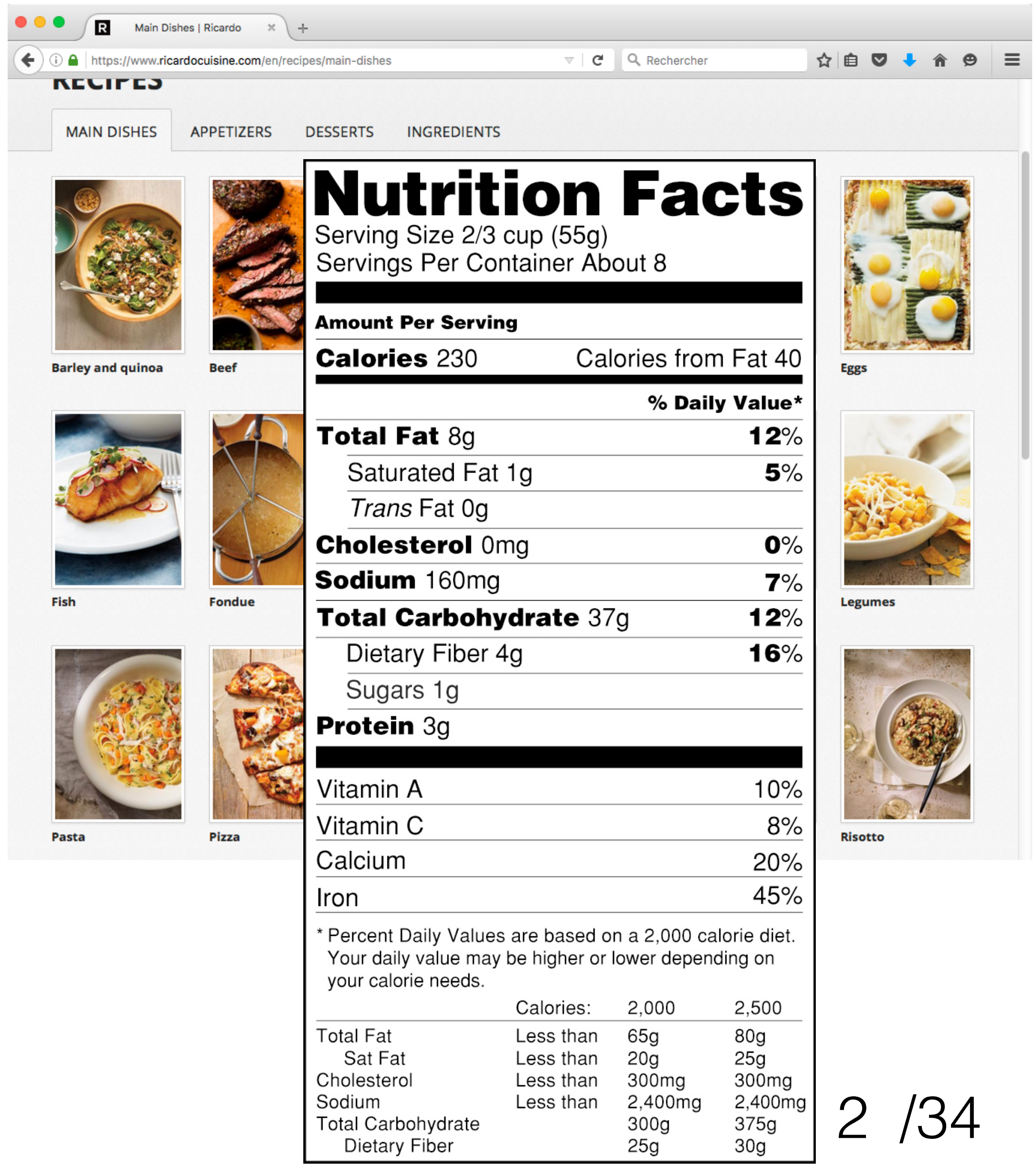
Canada Research Chairs   Chaires de recherche du Canada

Canada

The Diet Selection Problem

(Bertsimas & O'hair, 2013)

- 8.3% of U.S. population (25.8 million) have diabetes which can have serious complications
- Type II diabetes is known to be related to obesity
- Dietary change is among the most effective ways of preventing/controlling the disease
- A study made in the UK showed that only 20% of dieters last more than one month



Nutrition Facts
Serving Size 2/3 cup (55g)
Servings Per Container About 8

Amount Per Serving

Calories 230	Calories from Fat 40
	% Daily Value*
Total Fat 8g	**12**%
Saturated Fat 1g	**5**%
Trans Fat 0g	
Cholesterol 0mg	**0**%
Sodium 160mg	**7**%
Total Carbohydrate 37g	**12**%
Dietary Fiber 4g	**16**%
Sugars 1g	
Protein 3g	
Vitamin A	10%
Vitamin C	8%
Calcium	20%
Iron	45%

* Percent Daily Values are based on a 2,000 calorie diet. Your daily value may be higher or lower depending on your calorie needs.

	Calories:	2,000	2,500
Total Fat	Less than	65g	80g
Sat Fat	Less than	20g	25g
Cholesterol	Less than	300mg	300mg
Sodium	Less than	2,400mg	2,400mg
Total Carbohydrate		300g	375g
Dietary Fiber		25g	30g

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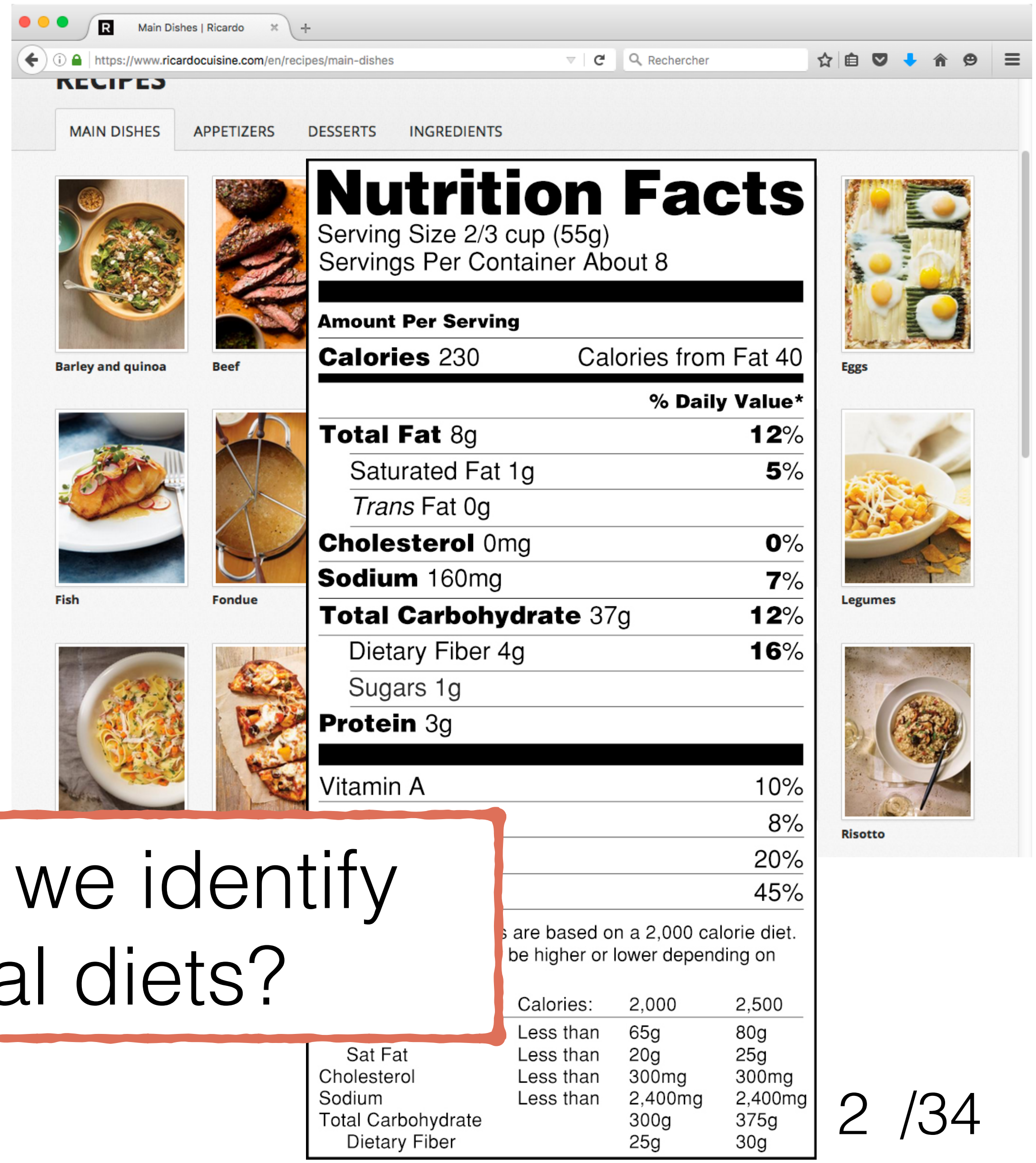

How can we identify optimal diets?

The Portfolio Selection Problem

- An individual meets with his financial advisor to tell him he wishes to invest in a given industrial sector, country, etc.
- Since uncertain factors affect performance, a « good » portfolio is one where the risks of losses are best justified by the potential gains

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How can we identify optimal investments?

Why are these decisions difficult to make?

- Something important is at stake
- Someone is held accountable for the decision
- The performance measure is multi-dimensional
- The alternatives are numerous

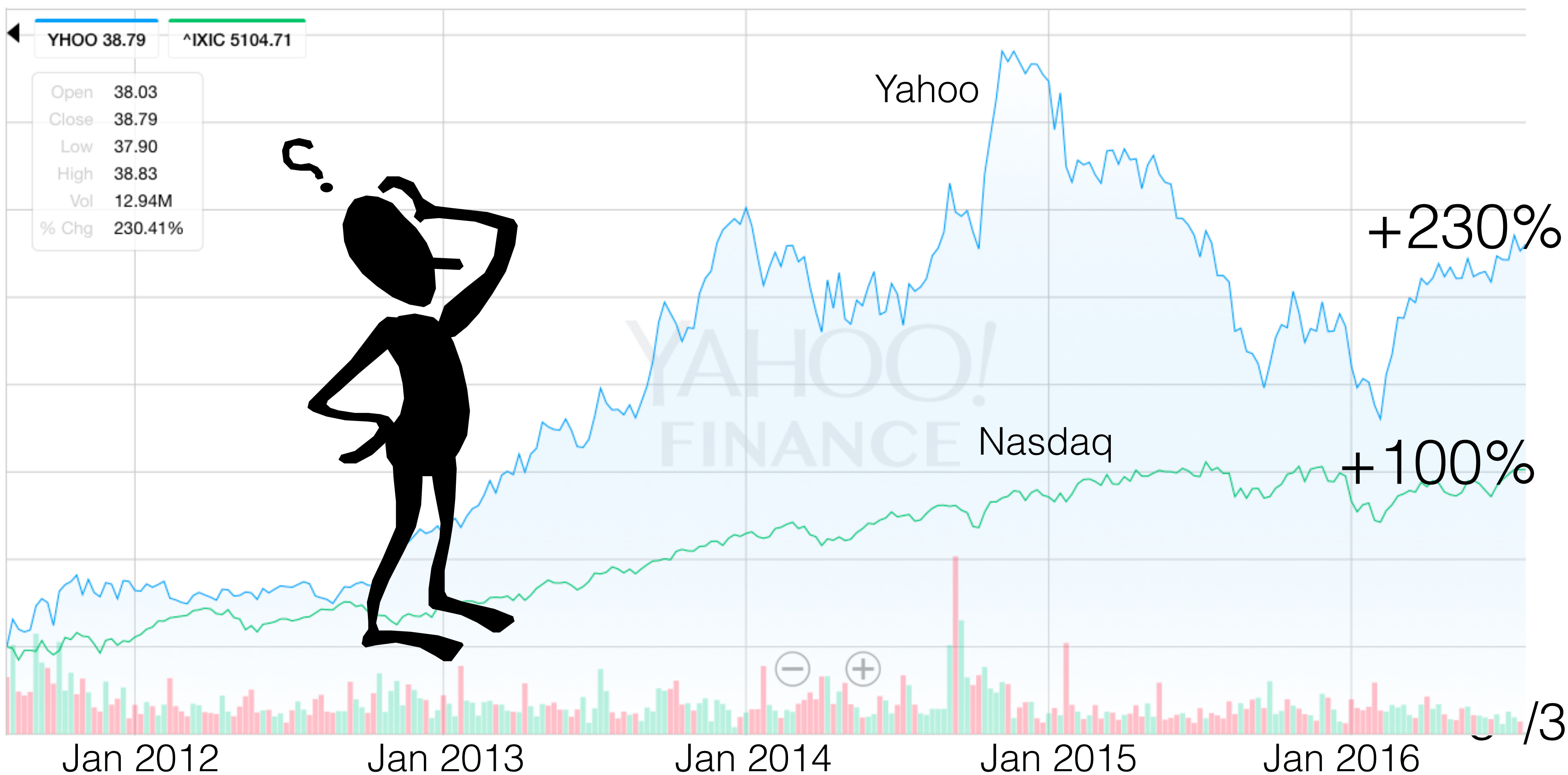
- Numerical optimization can only help once the decision maker’s subjective preferences have been fully characterized.

How can we elicit subjective preferences?

- An individual that wishes to control his diet can make pairwise comparisons of meals that he prefers

How can we elicit subjective preferences?

- An investor can indicate what type of wealth evolution he is comfortable with



The strength of utility theory

- In 1954, G. Debreu established that if the preference relation is complete, transitive, and continuous, then there exists a utility mapping such that
$$X \succeq Y \quad \Leftrightarrow \quad u(X) \geq u(Y)$$
where $X, Y \in \mathbb{R}^M$ describe two alternatives
- This implies that any such preference relation can be numerically optimized
$$\max_{x \in \mathcal{X}} \quad u(Z(x))$$

The limitations of utility theory

- Issue #1: One cannot sit down and make all possible comparisons

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- Solutions :
  - Make simplifying assumptions about the structure of u() in order to allow interpolation
  - Employ a scheme that handles uncertainty about u()

The limitations of utility theory

- Issue #2: One can easily provide false information about his preferences. (Kahneman & Tversky, 1979)

Example (adapted from Kahneman & Tversky, 1979)

We are expecting a new outbreak of the flu virus H1N1 next winter in Canada. Experts believe that 5000 people will die if nothing is done.

Which program should be adopted ?

Drug A (well studied): 4000 among those at risk would be saved
Drug B (experimental): 80% chance that everyone is saved, 20% that none are saved

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Drug C : 1000 people would end up dying

Drug D : 20% chance that all those at risk die, 80% that none of them die

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Drug B (experimental): 80% chance that everyone is saved, 20% that none are saved
Drug C **(same as A)**: 1000 people would end up dying
Drug D **(same as B)**: 20% chance that all those at risk die, 80% that none of them die

Many select Drug A & D but do not mean to be inconsistent.

The limitations of utility theory

- Issue #2: One can easily provide false information about his preferences. (Kahneman & Tversky, 1979)
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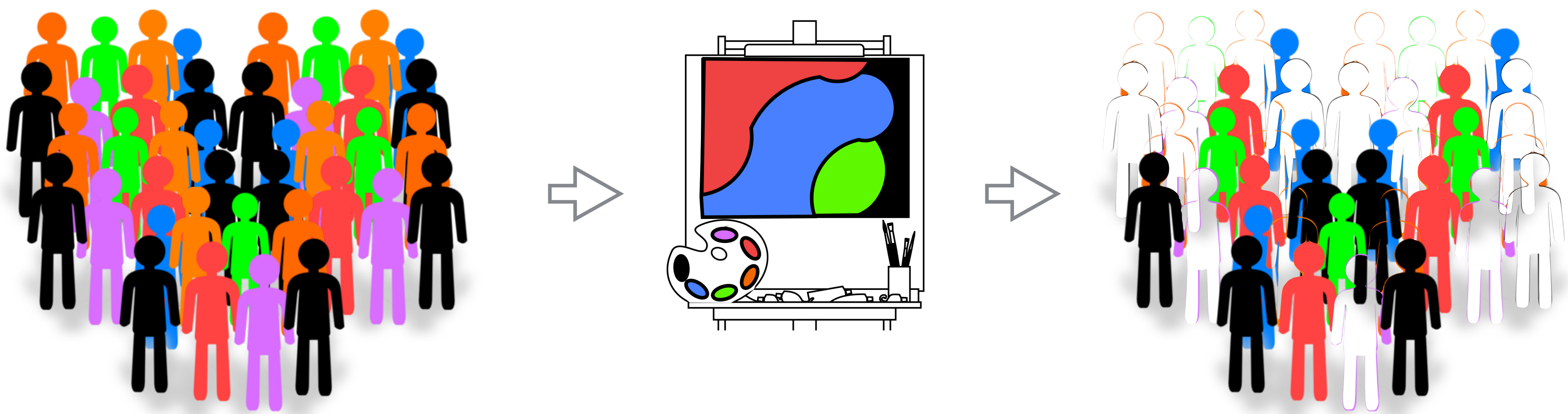
    - Make simplifying assumption about the structure of u() in order to filter the errors
    - Employ a scheme that handles uncertainty about u()

Uncertainty models for preferences

- Stochastic uncertainty: $u \sim F_u$
  (e.g. Chajewska et al. 2000)



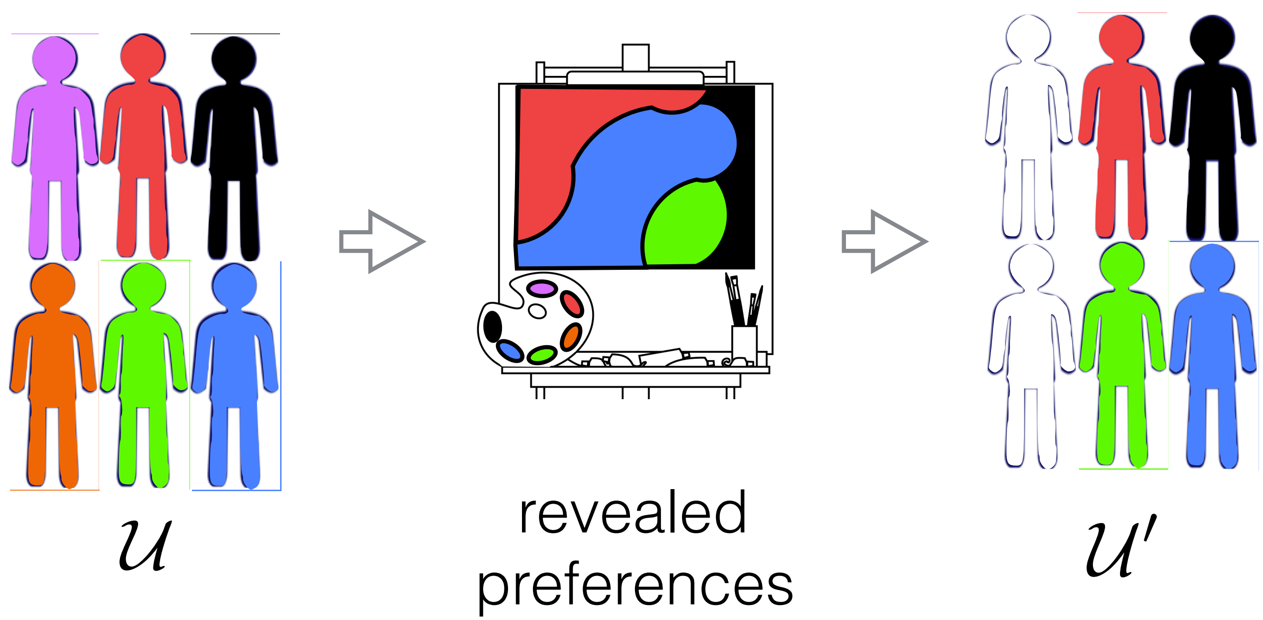
$u \sim F_u$

revealed preferences

$u \sim F'_u$

Uncertainty models for preferences

- Stochastic uncertainty: $u \sim F_u$
  (e.g. Chajewska et al. 2000)
- Knightian uncertainty: $u \in \mathcal{U}$
  (e.g. Hadar & Russell, 1969; Boutilier et al. 2006; Evren & Ok, 2011)



Preference robust optimization

- Max-min utility (Hu & Mehrotra, 2015):

$$\underset{x\in\mathcal{X}}{\text{maximize}} \qquad \min_{u\in\mathcal{U}} \; u(Z(x))$$

- Dominance constraint (Dentcheva & Ruszczynski, 2003; Haskell et al., 2016):

$$\begin{array}{ll} \underset{x\in\mathcal{X}}{\text{maximize}} & f(x) \\ \text{subject to} & u(Z(x)) \geq u(Z_0) \;,\; \forall\, u \in \mathcal{U} \end{array}$$

- Optimized dominance: choosing $\{Z_t\}_{t\in\mathbb{R}}\,,\; Z_t \succeq Z_{t'} \;,\; \forall\, t \geq t'$
(Armbruster & D., 2015; D. & Li, 2017)

$$\begin{array}{ll} \underset{x\in\mathcal{X},t\in\mathbb{R}}{\text{maximize}} & t \\ \text{subject to} & u(Z(x)) \geq u(Z_t) \;,\; \forall\, u \in \mathcal{U} \end{array}$$

The role of preference robust optimization in decision making under uncertainty

Decision making under uncertainty

- Let (Ω, Σ, P) be a probability space with $|\Omega| = M$ and $Z(x) : \Omega \to \mathbb{R}$ a random variable
- Preference between two R.V. depends on our attitude towards risk

Theorem. *Every decision maker expresses some level of aversion to risk*

Decision making under uncertainty

St. Petersburg paradox

I offer you to participate in the following game:

1. I flip an unbiased coin
2. First throw: if it falls on « head », I give you 2$, otherwise I flip again
3. Second throw: If it falls on « head », I give you 4$, otherwise I flip again
4. K-th throw: If it falls on « head », I give you 2^k\$, otherwise I flip again

What is the largest amount you would pay to play this game?

Decision making under uncertainty

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Wh

What if I tell you that the expected winning is infinite ?

$$\mathbb{E}[Z] = \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 4 + \cdots + \left(\frac{1}{2}\right)^k \cdot 2^k + \cdots = \infty$$

How can one assess risk tolerance?

[Grable & Lytton, Financial Services Review (1999)]

1. You have just finished saving for a « once-in-a-lifetime » vacation. Three weeks before you plan to leave, you lose your job. You would:
   A. Cancel the vacation
   B. Take a much more modest vacation
   C. Go as scheduled, reasoning that you need the time to prepare for a job search
   D. Extend your vacation, because this might be your last chance to go first-class

2. You are on a TV game show and can choose one of the following. Which would you take?
   A. $1,000 in cash
   B. A 50% chance at winning $ 5000
   C. A 25% chance at winning $ 10,000
   D. A 5% chance at winning $100,000

Axiomatic assumptions

- Monotonicity: $X \geq Y \;\Rightarrow\; X \succeq Y$
- Risk Aversion: $X \succeq Y \;\Rightarrow\; \theta X + (1-\theta)Y \succeq Y, \forall\, \theta \in [0,1]$
- Law Invariance: $X =_P Y \Rightarrow Y \approx X$
- Independence: $X \succeq Y \Rightarrow X$  $Z \succeq Y$  $Z, \forall\, Z$
- Translation Invariance: $X \succeq Y \Rightarrow X + t \succeq Y + t, \forall\, t$

- M+RA+LI+Indep. = Expected utility for some non-decreasing concave function (von Neumann & Morgenstern, 1944)
- M+RA+LI+TI = Law invariant convex risk measures: (Kusuoka, 2001)
  - Scale Invariance : $X \succeq Y \Rightarrow \alpha X \succeq \alpha Y, \forall\, \alpha \geq 0$

What are we left with?

- In the case of expected utility, let $\mathcal{S} := \mathbb{R}$:

$$\mathcal{U} := \{u \mid \exists\, v : \mathcal{S} \to \mathbb{R},\ u(\cdot) = \mathbb{E}[v(\cdot)],\ v(0) = 0$$

Monotonicity: $\exists\, \partial v : \mathcal{S} \to \mathbb{R},\ \partial v(\cdot) \geq 0$

Risk aversion: $v(y) \leq v(x) + (y - x)\partial v(x),\ \forall\, x, y \in \mathcal{S}$

Pairwise comparisons: $\mathbb{E}[v(W_k)] \geq \mathbb{E}[v(Y_k)],\ \forall\, k \quad \}$

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- Fortunately, we can replace $\mathcal{S}$ with finite set $\mathcal{S}'$ that only includes the joint support of W_k, Y_k and Z_t

What are we left with?

- In the case of risk measures, let $\mathcal{S} := \mathbb{R}^M$:

$$\mathcal{U} := \{u \mid \exists\, \rho : \mathcal{S} \to \mathbb{R},\ u(\cdot) = -\rho(\cdot),\ \rho(0) = 0$$

Monotonicity: $\exists\, \nabla\rho : \mathcal{S} \to \mathbb{R},\ \nabla\rho(\cdot) \leq 0$

Risk aversion: $\rho(Y) \geq \rho(X) + (Y - X)\nabla\rho(X),\ \forall\, X, Y \in \mathcal{S}$

Translation invariance: $1^T\nabla\rho(X) = -1,\ \forall\, X \in \mathcal{S}$

Scale invariance: $\rho(X) = X^T\nabla\rho(X),\ \forall\, X \in \mathcal{S}$

Law invariance: (see details in Delage & Li, 2016)

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How do we optimize ?

- Consider maximizing worst-case utility surplus:

$$\underset{x\in\mathcal{X}}{\text{maximize}} \qquad \psi(x) := \min_{u\in\mathcal{U}} \; u(Z(x)) - u(Z_t)$$

- One can actually show that

$$\psi(x) \;=\; \left\{ \begin{array}{l} \overbrace{\min_{u\in\mathcal{U}(\mathcal{S}')} \inf_{\alpha,\beta:\alpha^T X+\beta\leq u(X),\,\forall\, X\in\mathcal{S}'} \alpha^T Z(x) + \beta - u(Z_t)}^{\color{blue}{\text{Convex risk measure}}} \\ \\ \underbrace{\min_{u\in\mathcal{U}(\mathcal{S}')} \mathbb{E}[\inf_{\alpha,\beta:\alpha y+\beta\leq u(y),\,\forall\, y\in\mathcal{S}'} \alpha Z(x) + \beta - u(Z_t)]}_{\color{blue}{\text{Expected utility}}} \end{array} \right.$$

- Duality can be applied to get a finite dimensional optimization problem

How do we optimize ?

- Without further ado, here is the reformulation for expected utility (with $Z(x) := h(x, \xi(\omega))$):

$$
\begin{aligned}
&\underset{x\in\mathcal{X},\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)}}{\text{maximize}} && \nu_0 \\
&\text{Linear equalities} \left\{ \right. && \sum_i \mu_{i,j} - (\mathbb{P}(W_0 = \bar{y}_j) - \mathbb{P}(Y_0 = \bar{y}_j))\nu_0 \\
& && \quad -\sum_k (\mathbb{P}(W_k = \bar{y}_j) - \mathbb{P}(Y_k = \bar{y}_j))\nu_k \\
& && \quad +(\lambda_j^{(1)} - \lambda_{j-1}^{(1)}) - (\lambda_j^{(2)} - \lambda_{j-1}^{(2)}) \;=\; \mathbb{P}(Z_t = \bar{y}_j)\ \forall\, j \\
&\text{Linear inequalities} \left\{ \right. && \lambda_j^{(2)}(\bar{y}_{j+1} - \bar{y}_j) - \lambda_{j-1}^{(1)}(\bar{y}_j - \bar{y}_{j-1}) \leq 0\ \forall\, j \\
&\text{Convex constraints if h is concave} \left\{ \right. && \sum_j \bar{y}_j \mu_{i,j} \;\leq\; \mathbb{P}(\omega_i) h(x, \xi(\omega_i))\ \forall\, i \\
&\text{Linear equalities} \left\{ \right. && \sum_j \mu_{i,j} \;=\; \mathbb{P}(\omega_i)\ \forall\, i \\
& && \boldsymbol{\mu} \;\geq\; 0\,,\ \boldsymbol{\nu} \;\geq\; 0\,,\ \boldsymbol{\lambda}^{(1)} \;\geq\; 0\,,\ \boldsymbol{\lambda}^{(2)} \;\geq\; 0\,,
\end{aligned}
$$

How do we optimize ?

- Consider instead the case:

$$\begin{aligned} \underset{x \in \mathcal{X}, t \in \mathbb{R}}{\text{maximize}} \quad & t \\ \text{subject to} \quad & u(Z(x)) \geq u(Z_t) \ , \ \forall\, u \in \mathcal{U} \end{aligned}$$

- If t^* is known to lie in some interval $[t_0, t_1]$, then one can employ the bisection method on t:

$$\max_{t \in [t_0, t_1]} t : \max_{x \in \mathcal{X}} \min_{u \in \mathcal{U}} u(Z(x)) - u(Z_t) \geq 0$$

Value of optimized dominance model

1. Optimized dominance reduces to max-min utility when
$$u(Z_t) = u'(Z_t),\ \forall\, u, u' \in \mathcal{U},\ \forall\, t$$

   - In expected utility this occurs if $\mathcal{S} \subseteq [z_{\min},\ z_{\max}]$

$$Z_t := \begin{cases} z_{\max} & \text{with prob. } t \\ z_{\min} & \text{with prob. } 1-t \end{cases} \ \&\ \forall\, u \in \mathcal{U},\ \begin{Bmatrix} u(z_{\max}) = 1 \\ u(z_{\min}) = 0 \end{Bmatrix}$$

   - With convex risk measures, this occurs if
$$Z_t := t$$

Value of optimized dominance model

2. When $Z_t := t$, the proposed decision is guaranteed to be preferred to the largest available guaranteed return
   - As a consequence, it implements a conservative use of preference information

Value of optimized dominance model

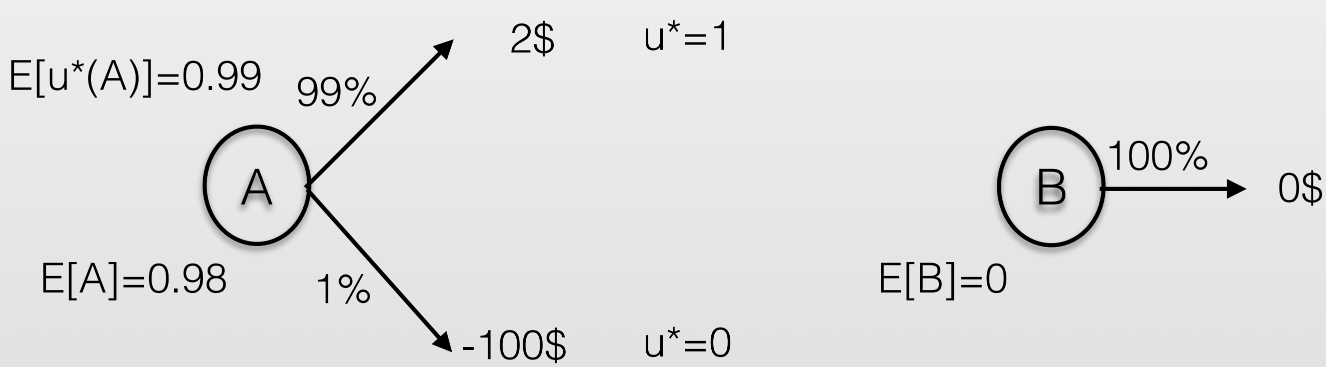
2.

<u>Example:</u>

Consider the following two risky investments.

$$\mathcal{U} := \{u \mid u \text{ concave},\ u(-100) = 0,\ u(2) = 1\}$$

Max-min expected utility would recommend :



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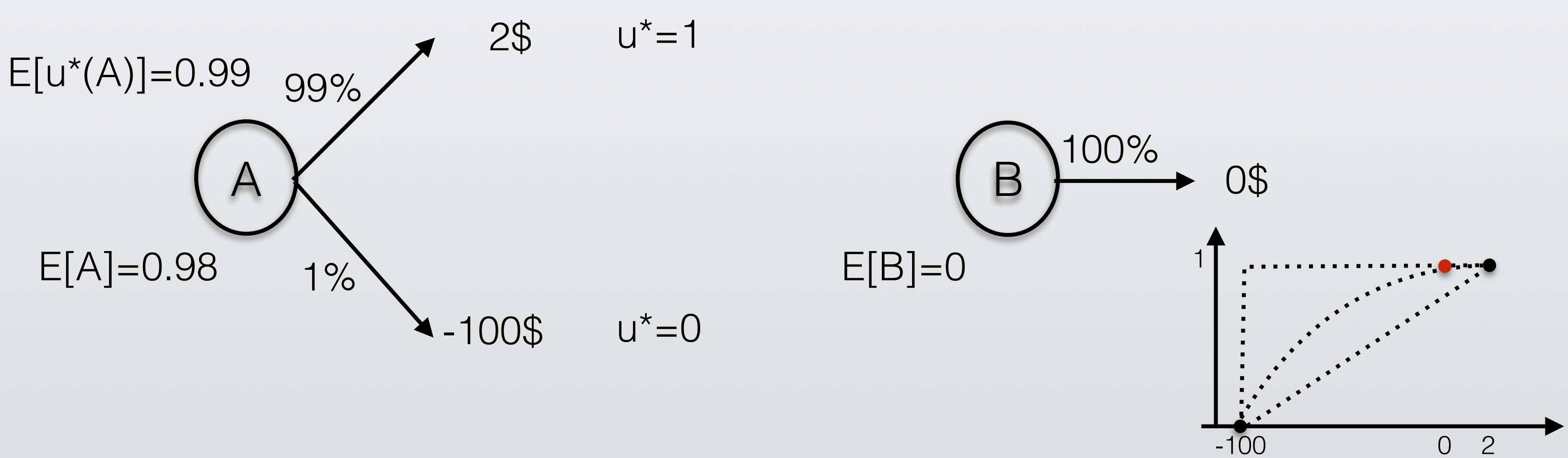

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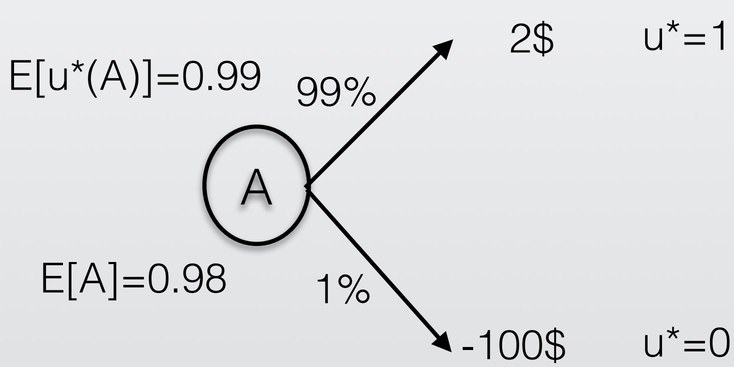

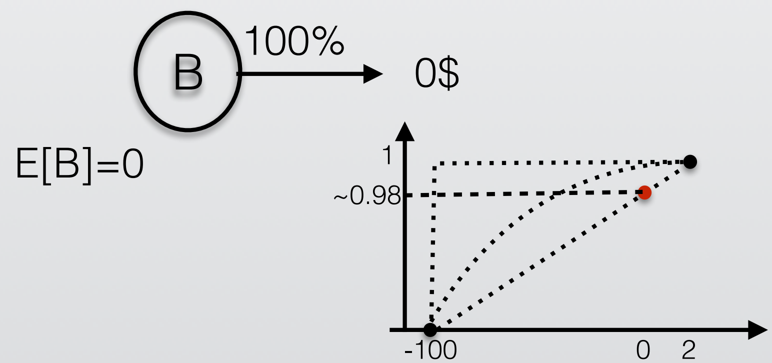

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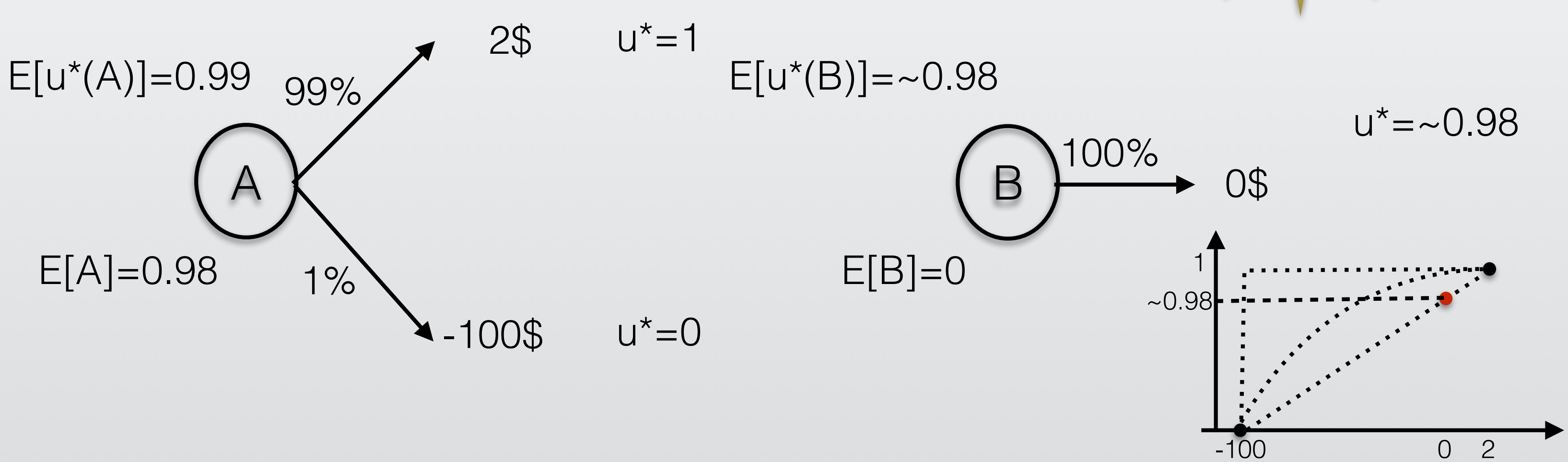

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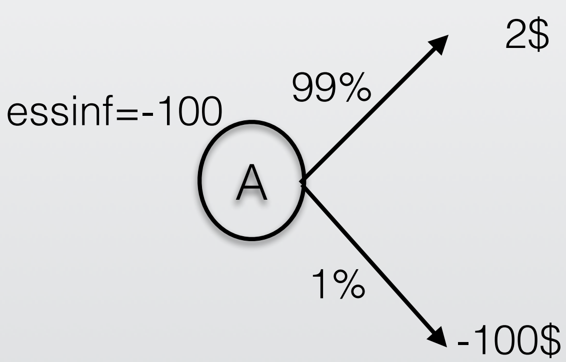
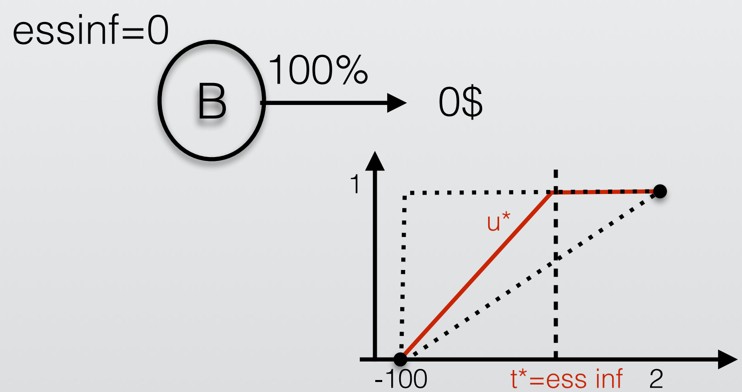
$$\Rightarrow \sup\{t : \min_{u\in\mathcal{U}} \mathbb{E}[u(Z(x)] - u(t) \geq 0\} = \operatorname{ess\,inf} Z(x)$$

Optimized dominance with $Z_t := t$ would recommend :





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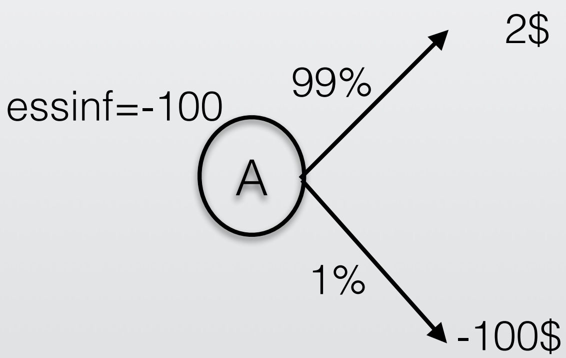
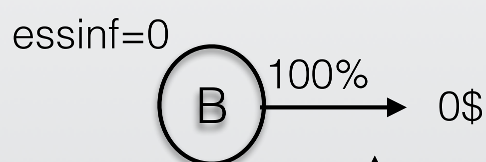
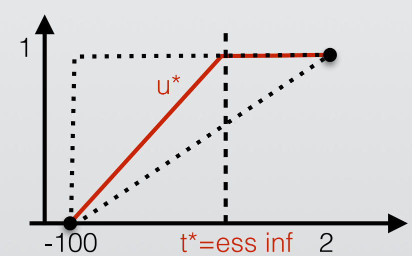
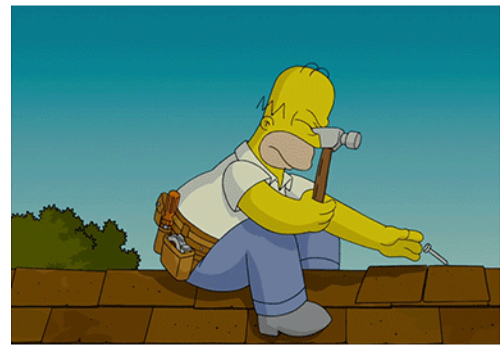






Accounting for elicitation errors

- Since even the best intention can lead to making mistakes, we might need to account for mislabeled comparisons



- One can replace the comparison constraint with:

$$\exists \delta \in \mathbb{R}^K,\ \|\delta\|_1 \leq \Gamma,\ u(W_k) + \delta_k \geq u(Y_k),\ \forall\, k = 1, \ldots, K$$

- Bertsimas and O'hair (2015) even propose accounting for some preference reversals with :

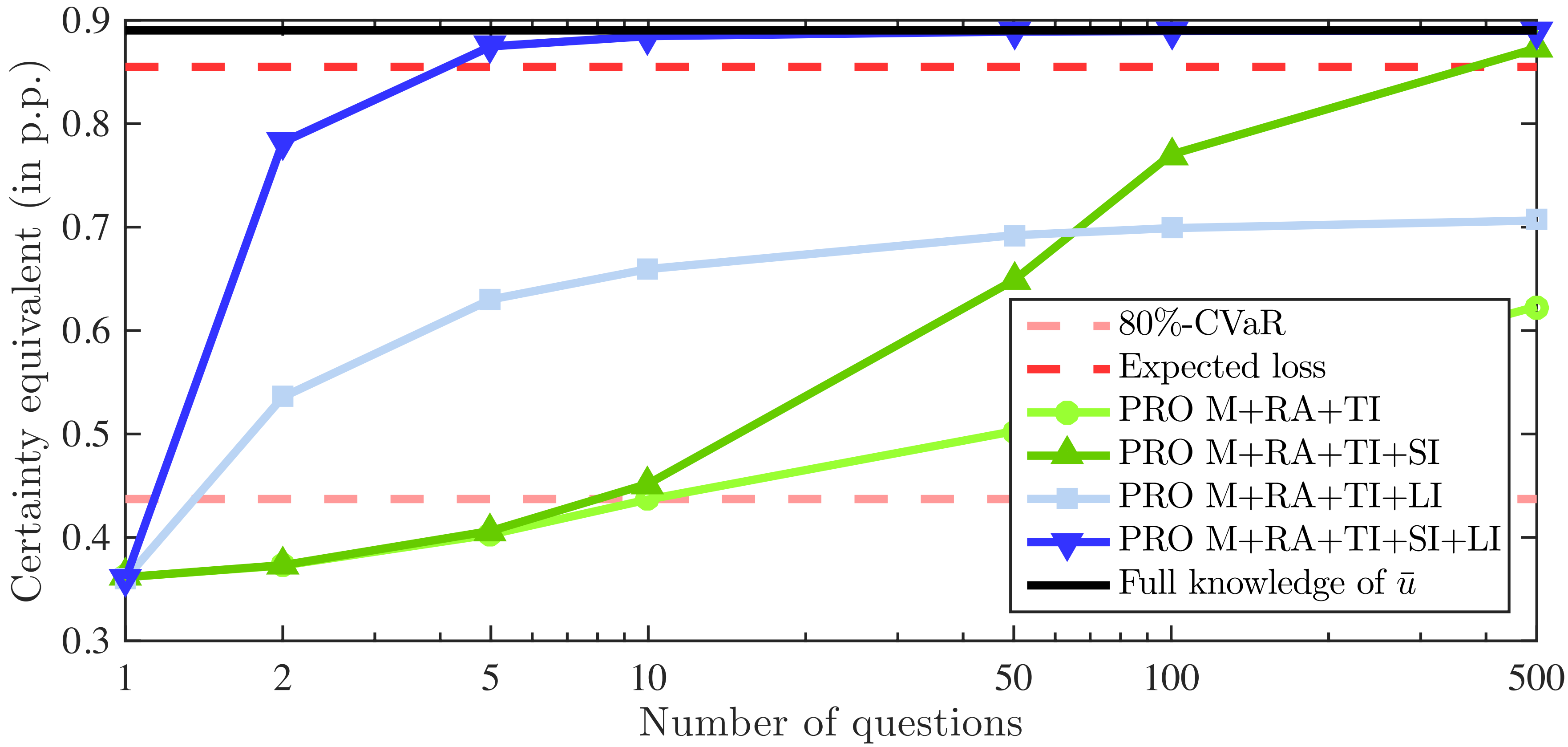
$$\exists z \in \{0,1\}^K,\ \sum z_k \leq \Gamma,\ \left\{ \begin{array}{c} u(W_k) \geq u(Y_k) - Mz_k \\ u(Y_k) \geq u(W_k) - M(1 - z_k) \end{array} \right\},\ \forall\, k$$

A brief survey of our numerical experiments

Numerical experiments

- Experiments are made using empirical stochastic models based on historical weekly returns from Yahoo Finance
- We create a synthetic decision maker with some choice of $\bar{u}(\cdot)$ which is kept hidden
- Information comes from a number of comparisons made using $\bar{u}(\cdot)$ for sets of randomly picked (W_k, Y_k)
- Results are averaged over a large number of stochastic models and sets of comparisons

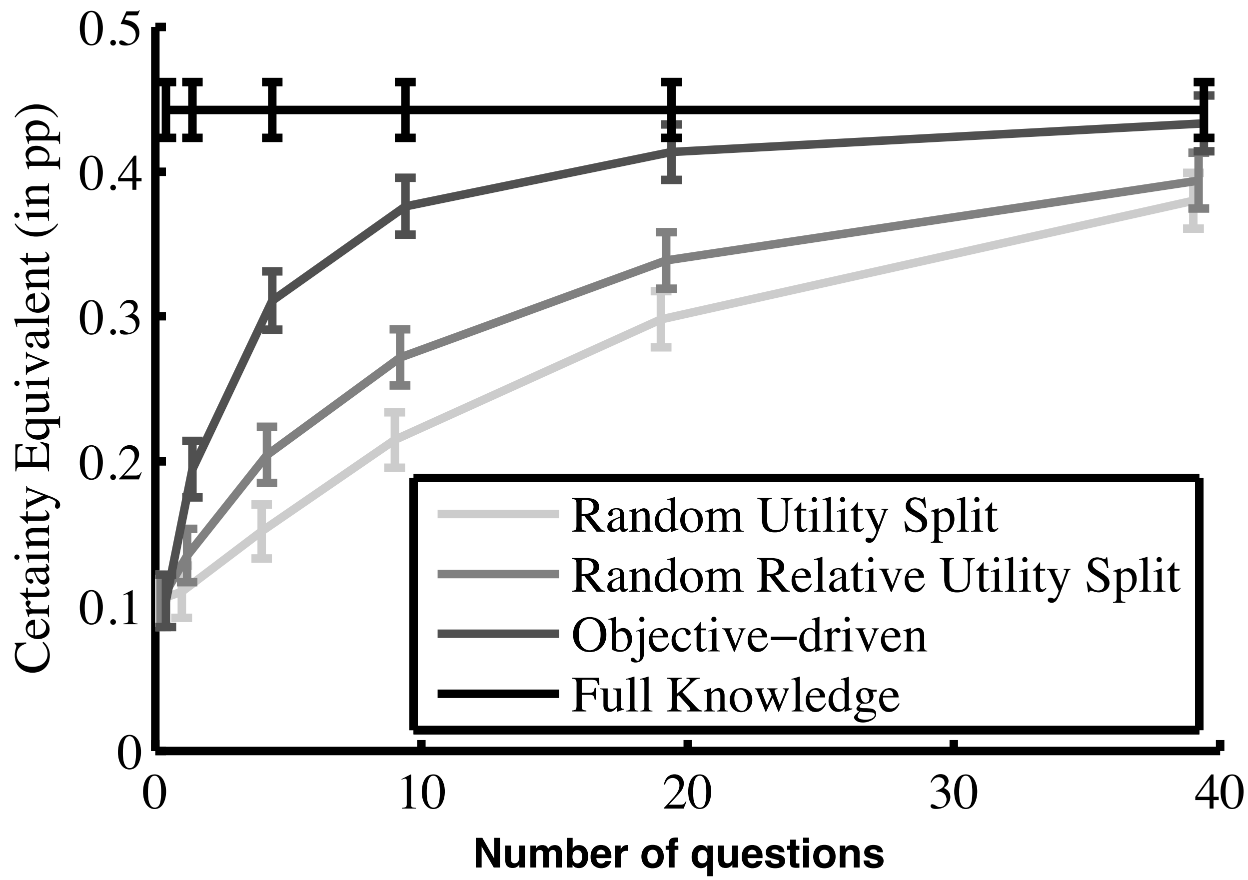
Performance in terms of certainty equivalent*



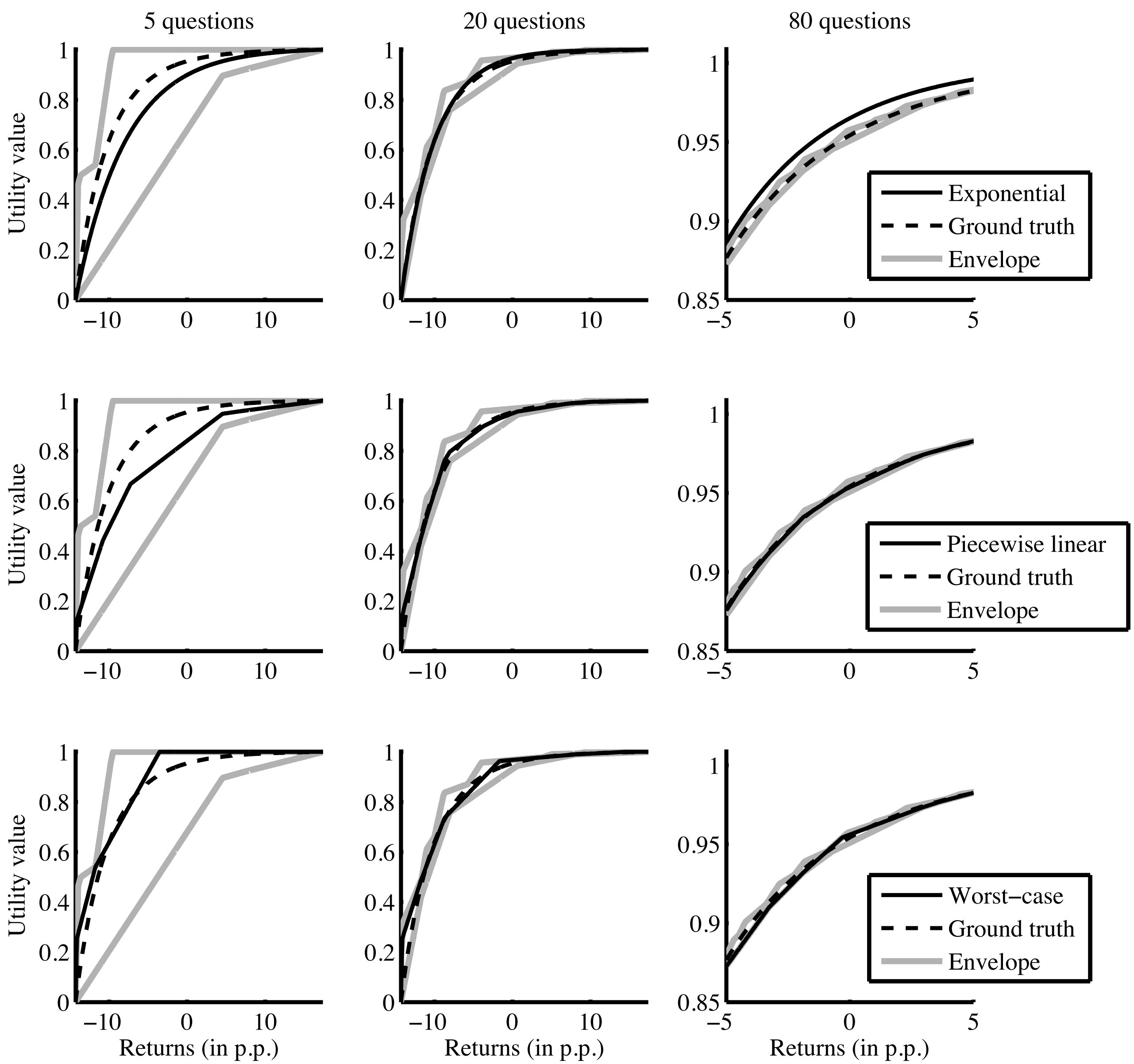
* Certainty equivalent : $\bar{u}(CE[Z(x)]) = \bar{u}(Z(x))$

Effect of elicitation strategy

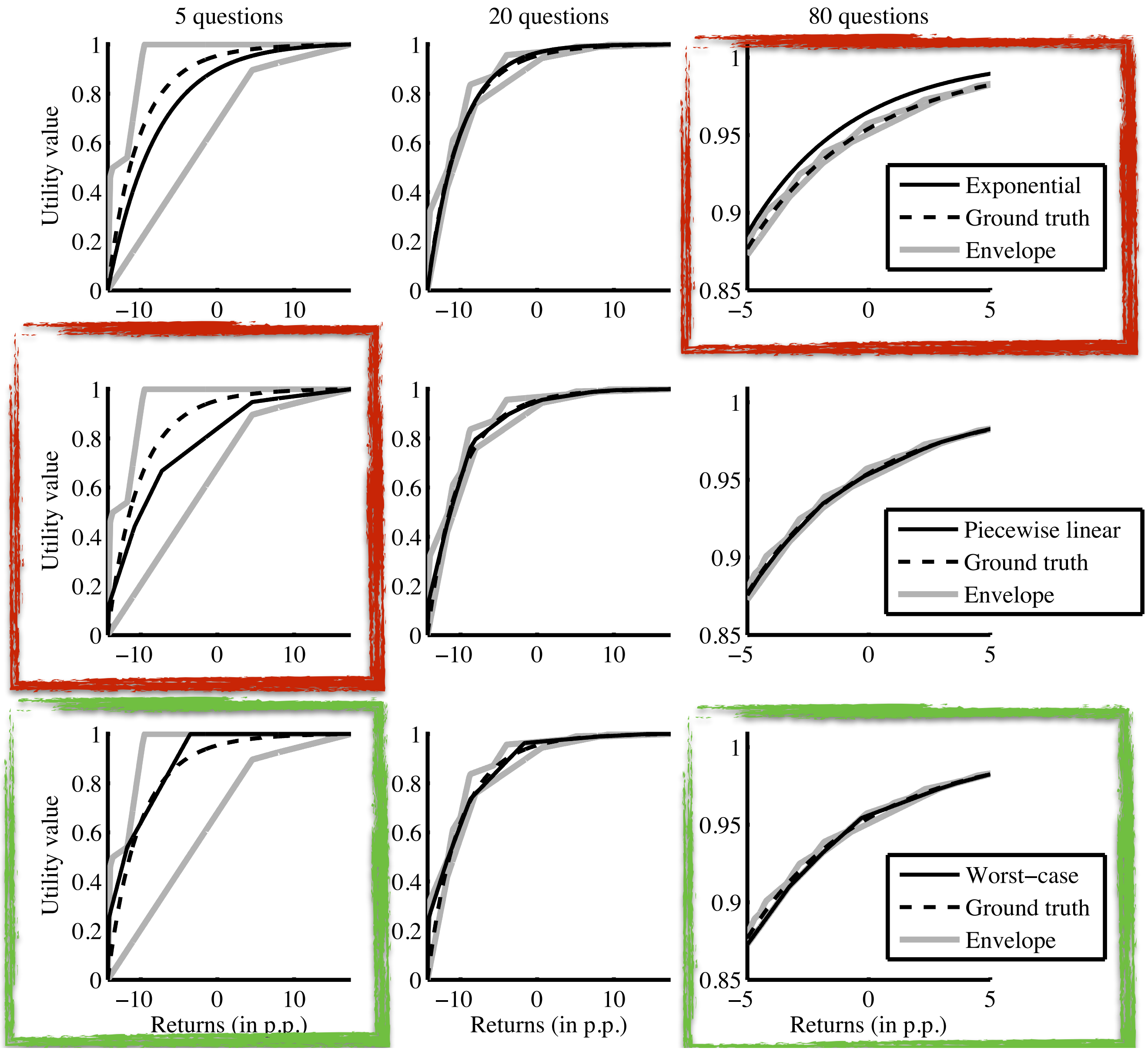
- One can improve convergence rate by designing effective elicitation strategies



Quality of recovered u()



Quality of recovered u()



- Exponential utility model does not recover true utility function
- Piecewise linear model takes excessive risks when information is limited
- Optimized dominance resolves both issues

Conclusion

- Many optimization problems involve objective functions that need to reflect the decision maker's preferences
- PRO accounts for the limited knowledge about these preferences : axioms + pairwise comparisons
- PRO offers valuable guarantees with respect to the unrevealed preference relation
- PRO preserves difficulty of resolution: LP —> LP
- Many potential applications: game theory, revenue management, expert systems, recommendation systems…

Thank you for your attention

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