Distributionally Robust Optimization under Moment Uncertainty with Application to Data-Driven Problems

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Uncertainty in Optimization

Consider an optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad h(x, \xi)$$

- x is the variable
- ξ is a vector of parameters

We often cannot assume to know ξ exactly and need ways to account for uncertainty:

- deterministic ξ estimated from noisy measurements
- ξ is actually a random parameter

Stochastic Programming

Assume $\xi \sim f_\xi$, solve :

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad \mathbb{E}_{f_\xi}[h(x, \xi)]$$

- Somewhat tractable (sample average approx.)
- Science goes into formulating distribution f_ξ

What if I don't know f_ξ?

- Access to limited information about f_ξ (e.g. samples)
- Future might not be distributed like the past
- Solution might be sensitive to my choice of f_ξ

Distributionally Robust Optimization

Describe uncertainty in f_ξ through a set $\mathcal{D}$ and consider worst case distribution:

$$(DRSP) \quad \underset{x\in\mathcal{X}}{\text{minimize}} \quad \max_{f_\xi\in\mathcal{D}} \; \mathbb{E}_{f_\xi}[h(x,\xi)] \;\; ,$$

In this talk, we consider that one can often describe his uncertainty in f_ξ through:

$$\mathcal{D}(\gamma) = \left\{ f_\xi \;\middle|\; \begin{array}{l} \mathbb{P}(\xi\in\mathcal{S}) = 1 \\ (\mathbb{E}[\xi]-\hat{\mu})^{\mathsf{T}}\hat{\Sigma}^{-1}(\mathbb{E}[\xi]-\hat{\mu}) \leq \gamma_1 \\ \mathbb{E}[(\xi-\hat{\mu})(\xi-\hat{\mu})^{\mathsf{T}}] \preceq (1+\gamma_2)\hat{\Sigma} \end{array} \right\} \;\; ,$$

for some $\gamma_1, \gamma_2 \geq 0$.

Related Work

- DRSP is related to the moment problem
  [Bertsimas & Popescu , 2005]
- If only support is known, then reduces to robust optimization
  [Ben-Tal & Nemirovski,1998]
- If mean & variance are known, special cases have been studied: e.g., newsvendor, portfolio selection, linear chance constraints, etc.
  [Scarf, 1958; Popescu, 2007; Calafiore & El Ghaoui, 2006]
- Prior to this work, little was known about how practical the DRSP is with $\mathcal{D}(\gamma)$

Outline

- Solving the Dist. Robust Stochastic Program
  - Conditions on $h(x, \xi)$ for existence of a tractable solution
- Using the DRSP in data-driven problems
  - Modeling the DRSP to get high confidence solutions
- Dist. Robust Portfolio Optimization
  - Empirical evaluation with real stock market data

Conditions on $h(x,\xi)$

Let $h(x,\xi) = \max_{k\in\{1,2,\ldots,K\}} h_k(x,\xi)$ such that for all k:

- $h_k(x,\xi)$ is convex in x
- $h_k(x,\xi)$ is concave in ξ
- $h_k(x,\xi)$'s value and gradient are easily obtained

Examples:

- $h(x,\xi) = -u(x^{\mathsf{T}}\xi)$ with $u(\cdot)$ piecewise linear concave
- Many $h(x,\xi)$ for which simpler robust form is tractable: i.e., $\text{minimize}_{x\in\mathcal{X}} \;\; \max_{\xi\in\mathcal{S}} \;\; h(x,\xi)$.

Solving Distrib. Robust Problems

Theorem 1. : *In the case that $h(x, \xi)$ satisfies our conditions, the following distributionally robust problem:*

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad \max_{f_\xi \in \mathcal{D}} \quad \mathbb{E}_{f_\xi}[h(x, \xi)]$$

1. *is NP-hard for general $\mathcal{D}$ (Bertsimas 2005)*
2. *can be solved in polynomial time for $\mathcal{D}(\gamma)$*
3. *stands with $\mathcal{D}(0)$ as a valuable relaxation of some NP-hard forms*

Solution Method

The following problem can be solved in polynomial time:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad \max_{f_\xi \in \mathcal{D}(\gamma)} \quad \mathbb{E}_{f_\xi}[h(x, \xi)]$$

After replacing $\max_{f_\xi \in \mathcal{D}(\gamma)} \mathbb{E}_{f_\xi}[h(x, \xi)]$ with its dual form,

$$\begin{array}{ll} \underset{x, \mathbf{Q}, \mathbf{q}, r}{\text{minimize}} & r + \left(\gamma_2 \hat{\Sigma} + \hat{\mu}\hat{\mu}^{\mathsf{T}}\right) \bullet \mathbf{Q} + \hat{\mu}^{\mathsf{T}}\mathbf{q} + \sqrt{\gamma_1}\, \|\hat{\Sigma}^{1/2}(\mathbf{q} + 2\mathbf{Q}\hat{\mu})\| \\ \text{subject to} & r \geq \max_{\xi \in \mathcal{S}} \left(h_k(x, \xi) - \xi^{\mathsf{T}}\mathbf{Q}\xi - \xi^{\mathsf{T}}\mathbf{q}\right) \quad , \quad \forall \;\; k \in \{1, ..., K\} \\ & \mathbf{Q} \succeq 0 \quad , \quad x \in \mathcal{X} \end{array}$$

and applying the ellipsoid method.

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Confidence region for f_ξ

Given that:

- I have M i.i.d. samples drawn from f_ξ
- I let $\hat{\mu}$ and $\hat{\Sigma}$ be empirical estimates
- I know that ξ lies in a ball of radius R

Then, based on concentration of f_ξ [McDiarmid, 1998], for some $\bar{\gamma}_1 = O\left(\frac{R^2}{M}\log(1/\delta)\right)$ and $\bar{\gamma}_2 = O\left(\frac{R^2}{\sqrt{M}}\sqrt{\log(1/\delta)}\right)$, we show that:

$$\mathbb{P}(f_\xi \in \mathcal{D}(\bar{\gamma})) \geq 1 - \delta \quad .$$

Solutions to Data-driven Problems

Given a set of samples $\{\xi_i\}_{i=1}^M$, build the distributional set $\mathcal{D}(\bar{\gamma})$ then, solve:

$$\underset{x\in\mathcal{X}}{\text{minimize}} \quad \max_{f_\xi\in\mathcal{D}(\bar{\gamma})} \quad \mathbb{E}_{f_\xi}[h(x,\xi)] \quad ,$$

Conclude that, with high probability, the solution has best worst case expected performance over a set of distributions that contains f_ξ.

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Portfolio Optimization

Stochastic Program form:

$$\underset{x \in \mathcal{X}}{\text{maximize}} \qquad \mathbb{E}_{f_\xi}[u(\xi^T x)]$$

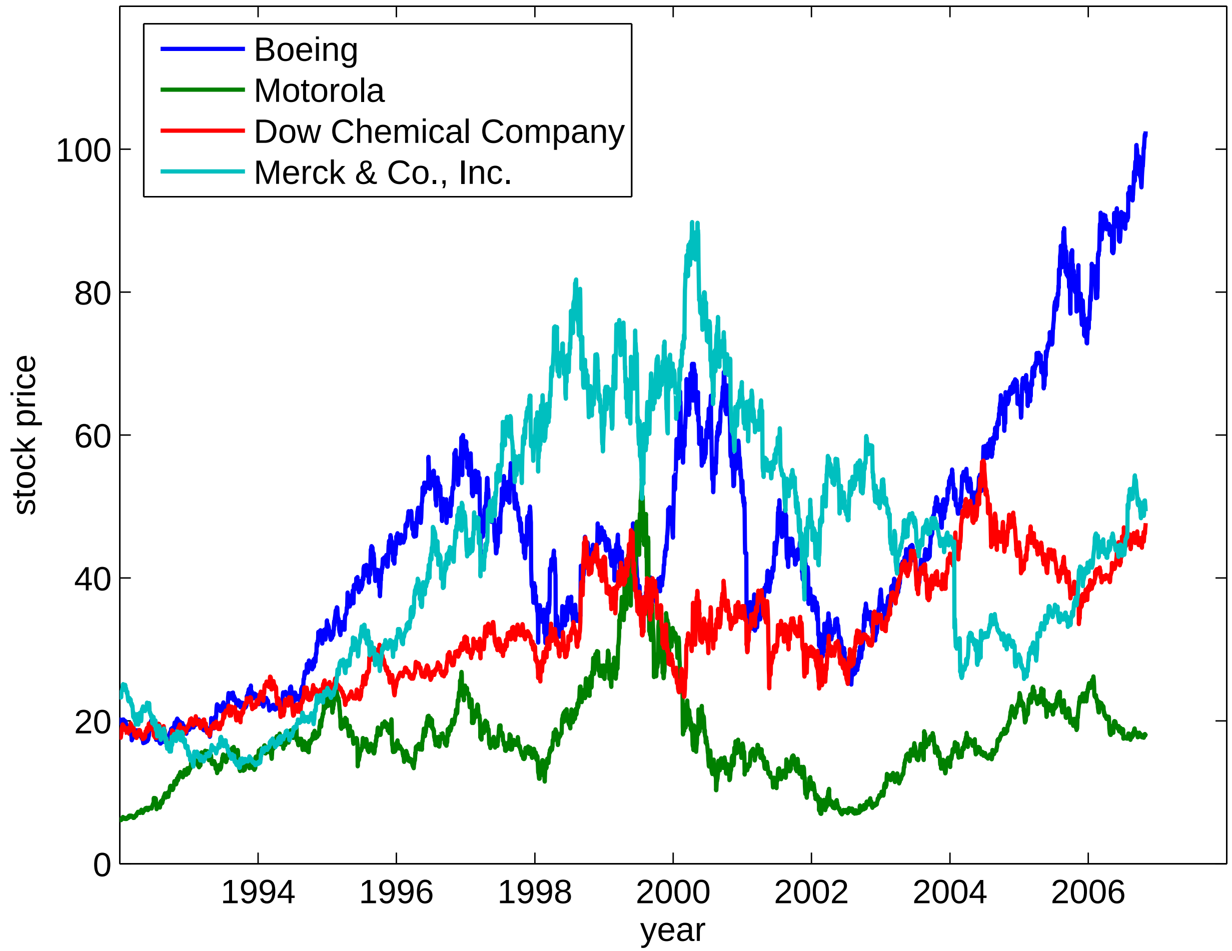
- x^j is how much do I invest in stock j
- Stock j returns in one day ξ^j for each dollar invested
- $u(\cdot)$ is a piecewise linear concave utility function

Distributionally Robust Portfolio Optimization form:

$$(DRPO) \qquad \underset{x \in \mathcal{X}}{\text{maximize}} \; \min_{f_\xi \in \mathcal{D}(\gamma)} \mathbb{E}_{f_\xi}[u(\xi^T x)]$$

Experiments with Historical Data

30 stocks were tracked over horizon (1992-2007)



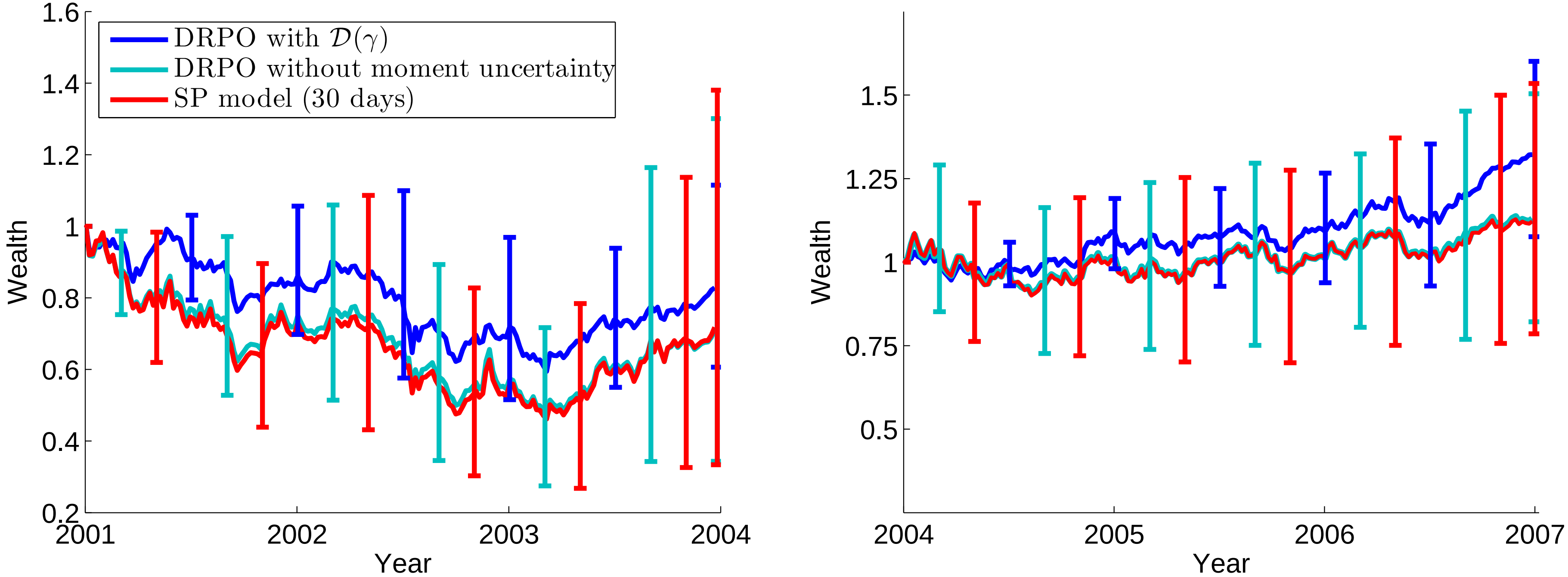
Experiments with Historical Data

An experiment consists of trading 4 stocks over (2001-07).

- Use (1992-2001) to choose γ_1 and γ_2
- Update portfolio on daily basis
- Estimate $\hat{\mu}$ and $\hat{\Sigma}$ based on a 30 days period
- DRPO with $\mathcal{D}(\gamma)$ is compared to :
  - DRPO without moment uncertainty
  - Stochastic Program using empirical distribution over last 30 days

Experimental Results

Comparison of wealth evolution in 300 experiments conducted over the years 2001-2007. For each model, the periodical 10% and 90% percentiles of wealth are indicated.



Summary

- Presented a DRSP model that accounts for uncertainty in the parameters of an optimization model
- Showed that this DRSP model is a natural one to use in data-driven problems
- Provided insights on how to formulate/recognize problems that are tractable
- Justified empirically the need to account for both distribution & moment uncertainty

- We encourage using a distributionally robust criterion as an objective or a constraint; hence, account for risks related to model ambiguity

Questions & Comments ...

... Thank you!

Experimental Results II

In finer details:

Method	2001-2004		2004-2007	
	Avg. yearly return	10-perc.	Avg. yearly return	10-perc.
DRPO model	0.944	0.846	1.1017	1.025
Popescu's DRPO model	0.700	0.334	1.047	0.9364
SP model	0.908	0.694	1.045	0.923

- 79% of the time, our DRPO outperformed both models
- On average accounting for moment uncertainty led to a relative gain of 1.67