Data-Driven Optimization with Distributionally Robust Second-Order Stochastic Dominance Constraints

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Stochastic Dominance

Definition (Second-Order Stochastic Dominance, SOSD)

Given any two random variables X and Y capturing some earnings, X stochastically dominates Y in the second-order, $\underline{X \succeq_{(2)} Y}$, if and only if

$$\int_{-\infty}^{\eta} F_X(t)\,dt \leq \int_{-\infty}^{\eta} F_Y(t)\,dt, \forall \eta \in \mathbb{R},$$

where $F_X(t) = \mathbb{P}(X \leq t)$.

Equivalent representations:

- $X \succeq_{(2)} Y \Leftrightarrow \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all non-decreasing concave functions u.
- $X \succeq_{(2)} Y \Leftrightarrow \mathbb{E}[(\eta - X)^+] \leq \mathbb{E}[(\eta - Y)^+], \forall \eta \in \mathbb{R}$.

Optimization with SOSD Constraints

Consider the SOSD Constrained Problem[1]:

$$\text{[SOSDCP]} \quad \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \ \boldsymbol{c}^\top \boldsymbol{x} \tag{1a}$$

$$\text{subject to} \ f(\boldsymbol{x},\boldsymbol{\xi}) \succeq_{(2)}^{\mathbb{P}} f_0(\boldsymbol{\xi}). \tag{1b}$$

- $f(\boldsymbol{x},\boldsymbol{\xi})$ is the random controlled performance function, and $f_0(\boldsymbol{\xi})$ is the random reference performance function: e.g., $f_0(\boldsymbol{\xi}) := f(\boldsymbol{x}_0,\boldsymbol{\xi})$ with $\boldsymbol{x}_0 \in \mathcal{X}$.

- E.g. SOSD constrained portfolio optimization problem:

$$\underset{\boldsymbol{x}:\ \mathbf{1}^\top\boldsymbol{x}=1,\boldsymbol{x}\geq 0}{\text{maximize}} \ \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}]^\top \boldsymbol{x}, \text{ s.t. } \boldsymbol{\xi}^\top \boldsymbol{x} \succeq_{(2)}^{\mathbb{P}} \boldsymbol{\xi}^\top \boldsymbol{x}_0.$$

[1] [Dentcheva and Ruszczyński. 2003]

Distributionally Robust Stochastic Dominance[2]

Definition (Distributionally Robust Second-Order Stochastic Dominance, DRSOSD)

Given two random variables X and Y, we say that X robustly stochastically dominates Y in the second order if and only if:

$$X \succeq_{(2)}^{\mathbb{P}} Y \quad \forall \mathbb{P} \in \mathcal{P},$$

where $X \succeq_{(2)}^{\mathbb{P}} Y$ refers to the fact that X stochastically dominates Y in the second-order when the probability measure is $\mathbb{P}$.

[2] [Dentcheva and Ruszczyński 2010]

Data-Driven DRSOSD using Wasserstein ambiguity

Consider DRSOSD constraint under a type-1 Wasserstein Ambiguity Set:

$$[\text{WDRSOSDCP}] \quad \underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \;\; \boldsymbol{c}^\top \boldsymbol{x} \tag{2a}$$

$$\text{subject to} \;\; f(\boldsymbol{x},\boldsymbol{\xi}) \succeq^{\mathbb{P}}_{(2)} f_0(\boldsymbol{\xi}) \qquad \forall \mathbb{P} \in \mathcal{P}^1_{\mathsf{W}}(\hat{\mathbb{P}},\epsilon), \tag{2b}$$

where $\hat{\mathbb{P}}$ is the empirical distribution of M i.i.d. observations.

Definition (Type-1 Wasserstein Ambiguity Set)

The type-1 Wasserstein ambiguity set[a] of radius ϵ centered at $\hat{\mathbb{P}}$ is defined by

$$\mathcal{P}^1_{\mathsf{W}}(\hat{\mathbb{P}},\epsilon) := \left\{ \mathbb{P} \in \mathcal{M}(\Xi) \,\middle|\, d^1_{\mathsf{W}}(\mathbb{P},\hat{\mathbb{P}}) \leq \epsilon \right\},$$

where $\mathcal{M}(\Xi)$ is the space of all distributions supported on Ξ and d^1_{W} is the Wasserstein metric.

[a] [Esfahani and Kuhn. 2018]

Special Cases of WDRSOSDCP

Proposition (Reduction to SOSDCP)

WDRSOSDCP with $\mathcal{P} := \mathcal{P}^1_W(\hat{\mathbb{P}}, 0)$ *reduces to SOSDCP with* $\mathbb{P} := \hat{\mathbb{P}}$.

Special Cases of WDRSOSDCP

Proposition (Reduction to SOSDCP)

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Proposition (Reduction to Robust Optimization)

WDRSOSDCP with $\mathcal{P} := \mathcal{P}^1_W(\hat{\mathbb{P}}, \infty)$ *reduces to the robust optimization problem,*

$$\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad \boldsymbol{c}^\top \boldsymbol{x} \tag{3a}$$

$$\text{subject to} \quad f(\boldsymbol{x}, \boldsymbol{\xi}) \geq f_0(\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \Xi. \tag{3b}$$

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What is the right extension when $\mathbb{P} \in \mathcal{P}$?

[Montes et al. 2014] propose six different extensions of SOSD:

$$X \succeq^{(4)} Y \quad \Leftrightarrow \quad F_X^{\mathbb{P}_1} \succcurlyeq_{(2)} F_Y^{\mathbb{P}_2},\ \forall \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P} \tag{4}$$

$$X \succeq^{(5)} Y \quad \Leftrightarrow \quad \exists \mathbb{P}_1 \in \mathcal{P},\ F_X^{\mathbb{P}_1} \succcurlyeq_{(2)} F_Y^{\mathbb{P}_2},\ \forall \mathbb{P}_2 \in \mathcal{P} \tag{5}$$

$$X \succeq^{(6)} Y \quad \Leftrightarrow \quad \forall \mathbb{P}_2 \in \mathcal{P},\ \exists \mathbb{P}_1 \in \mathcal{P},\ F_X^{\mathbb{P}_1} \succcurlyeq_{(2)} F_Y^{\mathbb{P}_2} \tag{6}$$

$$X \succeq^{(7)} Y \quad \Leftrightarrow \quad \exists \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P},\ F_X^{\mathbb{P}_1} \succcurlyeq_{(2)} F_Y^{\mathbb{P}_2} \tag{7}$$

$$X \succeq^{(8)} Y \quad \Leftrightarrow \quad \exists \mathbb{P}_2 \in \mathcal{P},\ F_X^{\mathbb{P}_1} \succcurlyeq_{(2)} F_Y^{\mathbb{P}_2},\ \forall \mathbb{P}_1 \in \mathcal{P} \tag{8}$$

$$X \succeq^{(9)} Y \quad \Leftrightarrow \quad \forall \mathbb{P}_1 \in \mathcal{P},\ \exists \mathbb{P}_2 \in \mathcal{P},\ F_X^{\mathbb{P}_1} \succcurlyeq_{(2)} F_Y^{\mathbb{P}_2}. \tag{9}$$

where:

$$F_1 \succcurlyeq_{(2)} F_2 \triangleq \int_{-\infty}^{\eta} F_1(t)\,dt \leq \int_{-\infty}^{\eta} F_2(t)\,dt, \forall \eta \in \mathbb{R}.$$

Recall, the extension from [Dentcheva and Ruszczyński 2010]:

$$\text{[DRSOSD]} \qquad X \succeq^{(*)} Y \Leftrightarrow F_X^{\mathbb{P}} \succcurlyeq_{(2)} F_Y^{\mathbb{P}},\ \forall \mathbb{P} \in \mathcal{P}.$$

Axiomatic Motivation for DRSOSD

- Consider a non-atomic ambiguous probability space $(\Omega, \mathcal{F}, \mathcal{P})$.
- Let $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) = \cap_{\mathbb{P}\in\mathcal{P}} \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathbb{P})$
- Let $\mathcal{U} := \{X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P}) \mid \exists F_X, F_X = F_X^{\mathbb{P}}, \forall \mathbb{P} \in \mathcal{P}\}$
- Non-atomic $\Rightarrow \forall X, \forall \mathbb{P} \in \mathcal{P}, \exists X^{\mathbb{P}} \in \mathcal{U}, F_{X^{\mathbb{P}}} = F_X^{\mathbb{P}}$

Theorem

If the preference relation $\succeq$ satisfies:

- *(SOSD on $\mathcal{U}$) If $\{X, Y\} \subset \mathcal{U}$, then $X \succeq Y \Leftrightarrow F_X \succeq_{(2)} F_Y$.*
- *(Ambiguity Monotonicity) If $X^{\mathbb{P}} \succeq Y^{\mathbb{P}}$ for all $\mathbb{P} \in \mathcal{P}$, then $X \succeq Y$*
- *(Maximal Ambiguity Indecisiveness) If $\exists \mathbb{P} \in \mathcal{P}$ such that $X^{\mathbb{P}} \not\succeq Y^{\mathbb{P}}$, then $X \not\succeq Y$.*

Then, for any random variables $X, Y \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P})$, we have that $X \succeq Y$ if and only if $X \succeq^{(*)} Y$, i.e. $X \succeq_{(2)}^{\mathbb{P}} Y \quad \forall \mathbb{P} \in \mathcal{P}$.

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Data-Driven WDRSOSDCP

- Assumption 1: The feasible set $\mathcal{X}$ is a non-empty convex set and the outcome space Ξ is a non-empty compact convex set.
- Assumption 2: $f(\boldsymbol{x}, \boldsymbol{\xi})$ and $f_0(\boldsymbol{\xi})$ are piecewise linear concave in $\boldsymbol{x}$ and $\boldsymbol{\xi}$.
- Assumption 3: $\mathcal{P}^1_{\mathsf{W}}(\hat{\mathbb{P}}, \epsilon)$ uses the ℓ_1-norm or ℓ_∞-norm as the reference metric.

Finite sample guarantee of WDRSOSDCP solutions

Proposition

Suppose that Assumption 1 holds and that each observations in $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^M$ *are drawn i.i.d. from some* $\bar{\mathbb{P}}$*, with* $M \geq 1$ *and* $m > 2$*. Given some* $\beta \in (0,1)$*, let* $\hat{\boldsymbol{x}}_M$ *be the optimal solution of the WDRSOSDCP with ambiguity set* $\mathcal{P}_W^1(\hat{\mathbb{P}}, \epsilon_M(\beta))$ *where*

$$\epsilon_M(\beta) := \begin{cases} \left(\frac{\log(c_1\beta^{-1})}{c_2 M}\right)^{1/\max(m,2)} & \text{if } M \geq \frac{\log(c_1\beta^{-1})}{c_2} \\ \left(\frac{\log(c_1\beta^{-1})}{c_2 M}\right)^{1/a} & \text{otherwise}, \end{cases}$$

and where c_1*,* c_2*, and* $a > 1$ *are positive constants (see [Esfahani and Kuhn. 2018] for details). One has the guarantee that, with probability larger than* $1-\beta$*,* $\hat{\boldsymbol{x}}_M$ *satisfies the SOSD constraint under* $\bar{\mathbb{P}}$*, i.e.,* $f(\hat{\boldsymbol{x}}_M, \boldsymbol{\xi}) \succeq_{(2)}^{\bar{\mathbb{P}}} f_0(\boldsymbol{\xi})$*.*

Asymptotic consistency of WDRSOSDCP solutions

Proposition

Suppose that assumptions 1 and 2 hold, that $\mathcal{X}$ is bounded, and that $\beta_M \in (0,1)$ satisfies $\sum_{M=1}^{\infty} \beta_M < \infty$ and $\lim_{M\to\infty} \epsilon_M(\beta_M) = 0$. Consider

[ϕ-SOSDCP]
$$\begin{aligned} &\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \;\; \boldsymbol{c}^\top \boldsymbol{x} \\ &\text{subject to} \;\; \mathbb{E}_{\bar{\mathbb{P}}}\left[(t - f(\boldsymbol{x},\boldsymbol{\xi}))^+\right] \le \mathbb{E}_{\bar{\mathbb{P}}}\left[(t - f_0(\boldsymbol{\xi}))^+\right] + \phi \quad \forall t \in \mathbb{R}, \end{aligned}$$

with $\phi > 0$, and assume that Slater's condition is satisfied. Let $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^M$ be i.i.d. from $\bar{\mathbb{P}}$, $\boldsymbol{x}_M$ be an optimal solution of:

$$\begin{aligned} &\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \;\; \boldsymbol{c}^\top \boldsymbol{x} \\ &\text{subject to} \;\; \mathbb{E}_{\mathbb{P}}\left[(t - f(\boldsymbol{x},\boldsymbol{\xi}))^+\right] \le \mathbb{E}_{\mathbb{P}}\left[(t - f_0(\boldsymbol{\xi}))^+\right] + \phi \quad \left\{ \begin{array}{c} \forall t \in \mathbb{R} \\ \forall \mathbb{P} \in \mathcal{P}_W^1(\hat{\mathbb{P}}, \epsilon_M(\beta_M)) \end{array} \right. \end{aligned}$$

and $\mathcal{X}^*$ be the set of optimal solutions to the ϕ-SOSDCP under the true distribution $\bar{\mathbb{P}}$. Then $\boldsymbol{x}_M$ converges almost surely to $\mathcal{X}^*$ as M goes to infinity.

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Optimization with DRSOSD Constraints

- Relevant studies (continue):

- [Guo et al. 2017]: use a discretization scheme to approximate DRSOSD constrained problem under a moment-based ambiguity set.
- [Kozmík. 2019]: study a portfolio optimization problem with DRSOSD constraints under type-1 Wasserstein ball over the space of M-points distributions, and derive a conservative approximation.
- [Sehgal and Mehra. 2020]: study a robust portfolio optimization problem with SOSD where scenario perturbations lie within a budgeted uncertainty set.
- [Mei et al. 2022] study independently the WDRSOSDCP and propose a novel split-and-dual decomposition framework

Multistage Robust Optimization Reformulation

Proposition

WDRSOSDCP with $\epsilon \in (0, \infty)$ coincides with the optimal value of the following multistage robust optimization problem:

$$
\begin{aligned}
\underset{\boldsymbol{x}\in\mathcal{X}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} \quad & L(\boldsymbol{x}, t) \le 0 & \forall t \in \bar{\mathcal{T}}, \\
\textit{where} \quad & L(\boldsymbol{x}, t) := \inf_{\lambda, \boldsymbol{q}} \lambda\epsilon + \frac{1}{M}\sum_{i=1}^{M} q_i & \\
\textit{s.t.} \quad & g(\boldsymbol{x}, \boldsymbol{\xi}, t) - \lambda\|\boldsymbol{\xi} - \hat{\xi}_i\| \le q_i & \forall i \in [M], \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{\lambda} \ge 0, \boldsymbol{q} \in \mathbb{R}^M, &
\end{aligned}
$$

where $g(\boldsymbol{x}, \boldsymbol{\xi}, t) := (t - f(\boldsymbol{x}, \boldsymbol{\xi}))^+ - (t - f_0(\boldsymbol{\xi}))^+$ *and* $\bar{\mathcal{T}} := \left[\inf_{\boldsymbol{\xi}\in\Xi} f_0(\boldsymbol{\xi}), \sup_{\boldsymbol{\xi}\in\Xi} f_0(\boldsymbol{\xi})\right]$.

- Multistage robust optimization problem: $\min_{\boldsymbol{x}}\text{-}\sup_{t}\text{-}\inf_{\boldsymbol{\lambda},\boldsymbol{q}}\text{-}\sup_{\boldsymbol{\xi}}$.

- Multistage robut linear optimization problem under Assumption 3 and when $\mathcal{X}$ and Ξ are polyhedral.

An Exact Solution Scheme

Inspired by [Postek and Hertog. 2016] and [Bertsimas and Dunning. 2016]

Algorithm Iterative Partition based Solution Algorithm

Initialize: $\text{LB}^0 = -\infty$, $\text{UB}^0 = +\infty$, $\ell = 1$, $\hat{\mathcal{T}}^0 := \emptyset$, ε.
Initialize: $\mathscr{P}^1 := \{\bar{\mathcal{T}}\}$, $\bar{\mathcal{T}} := [\inf_{\boldsymbol{\xi}\in\Xi} f_0(\boldsymbol{\xi}), \sup_{\boldsymbol{\xi}\in\Xi} f_0(\boldsymbol{\xi})]$.
while $|(\text{UB}^{\ell-1} - \text{LB}^{\ell-1})/\text{UB}^{\ell-1}| > \varepsilon$ **do**
  Solve an upper bound problem with the partition $\mathscr{P}^\ell$ and linear decision rules.
  Identify the optimal solution $(\boldsymbol{x}^{*\ell}, \boldsymbol{\lambda}^{*\ell}, \boldsymbol{q}^{*\ell}, \bar{\boldsymbol{q}}^{*\ell})$ and optimal objective UB^ℓ.
  Calculate an active scenarios set $\hat{\mathcal{A}}^\ell$.
  Construct the finite scenarios set $\hat{\mathcal{T}}^\ell \leftarrow \hat{\mathcal{A}}^\ell \bigcup \hat{\mathcal{T}}^{\ell-1}$.
  Solve a lower bound problem with $\hat{\mathcal{T}}^\ell$ and identify the new LB^ℓ.
  Update the partitions $\mathscr{P}^{\ell+1} \leftarrow \mathcal{V}(\mathscr{P}^\ell, \hat{\mathcal{A}}^\ell)$, and $\ell := \ell + 1$.
end while
return optimal objective value z^* and optimal solution $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{q}^*, \bar{\boldsymbol{q}}^*)$.

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Application to Portfolio Optimization

DRSOSD constrained portfolio optimization problem with uncertain returns $\boldsymbol{\xi}$:

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathcal{X}}{\text{maximize}} \;\; & \mathbb{E}_{\hat{\mathbb{P}}}[\boldsymbol{\xi}]^\top \boldsymbol{x} \\ \text{subject to} \;\; & \boldsymbol{\xi}^\top \boldsymbol{x} \succeq_{(2)}^{\mathbb{P}} \boldsymbol{\xi}^\top \boldsymbol{x}_0 \qquad \forall \mathbb{P} \in \mathcal{P}^1_{\mathsf{W}}(\hat{\mathbb{P}}, \epsilon), \end{aligned}$$

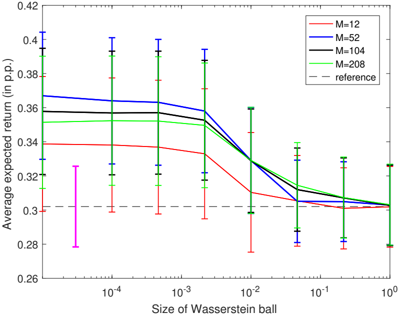
where $\mathcal{X} := \left\{ \boldsymbol{x} \in \mathbb{R}^m \;\middle|\; \sum_{j=1}^{m} x_j = 1, x_j \geq 0, \forall j \in \{1, \cdots, m\} \right\}$.

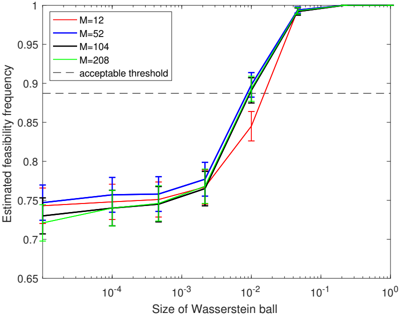
- $\boldsymbol{x}_0$ is a reference portfolio.
- Assume Ξ to be a box, i.e. $\Xi := \{\boldsymbol{\xi}^- \leq \boldsymbol{\xi} \leq \boldsymbol{\xi}^+\}$.

We experiment with both synthetic data and real stock market data.

Real Stock Data: Calibration of M and ϵ

- In-sample data: Weekly stock returns from 335 companies from S&P 500 over Jan 1994 - Dec 2013
- Portfolio optimization over 5 randomly picked stocks with x_0 as the equally weighted portfolio
- Cross-validation of look-back period and Wasserstein ball size based on distribution of next 26 weekly returns





Real Stock Data: Out-of-sample performance

- Out-of-sample data: Weekly stock returns from 257 companies from S&P 500 over Jan 2014 - Dec 2019
- SOSDCP uses lookback of $M = 52$ weeks
- WDRSOSDCP uses $M = 52$ and $\epsilon = 0.01$
- Performance is averaged over 1000 runs.

Descriptive statistics (in p.p.)	SOSDCP	WDRSOSDCP	Reference	Acc. thresh.
Average expected return	0.183	0.190	0.184	-
Average standard deviation	0.032	0.029	0.022	-
Average CVaR (conf. 90%)	0.054	0.048	0.037	-
SOSD feasibility frequency	87%	96%	100%	94%

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Concluding Remarks

- We provide an axiomatic motivation for distributionally robust version of the second-order stochastic dominance ordering
- We show that the data-driven WDRSOSDCP can provide finite sample guarantees and asymptotic consistency
- We develop an efficient exact solution scheme, an iterative partition-based solution algorithm.
- We show how out-of-sample SOSD feasibility can be improved by carefully adjusting the level of robustification without sacrificing much objective perfomance.

Questions & Comments...

Our paper is available on Optimization Online via

...Thank you!

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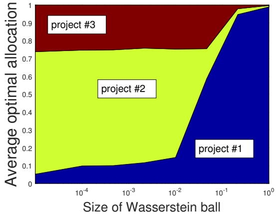
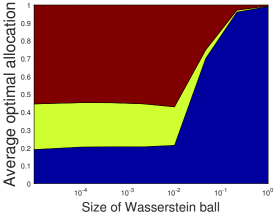
Synthetic Data: Optimal Allocation

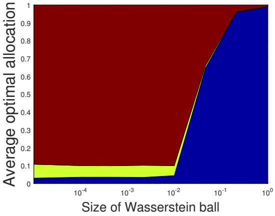
- Consider 3 assets (asset #1, #2 and #3) that are independent from each other, while their respective marginal distribution of return is such that $\xi_3 \succ^{\bar{\mathbb{P}}}_{(2)} \xi_1 \succ^{\bar{\mathbb{P}}}_{(2)} \xi_2$ and $\mathbb{E}_{\bar{\mathbb{P}}}[\xi_3] > \mathbb{E}_{\bar{\mathbb{P}}}[\xi_2] = \mathbb{E}_{\bar{\mathbb{P}}}[\xi_1]$.
- $\boldsymbol{x}_0 := [1, 0, 0]$, invest all the resources in project #1.
- Generate $M \in \{10, 100, 1000\}$ i.i.d. in-sample data.
- Results are averaged over 100 runs.



10 samples

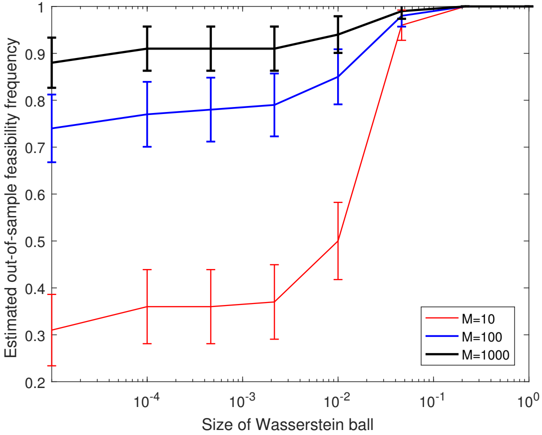


100 samples



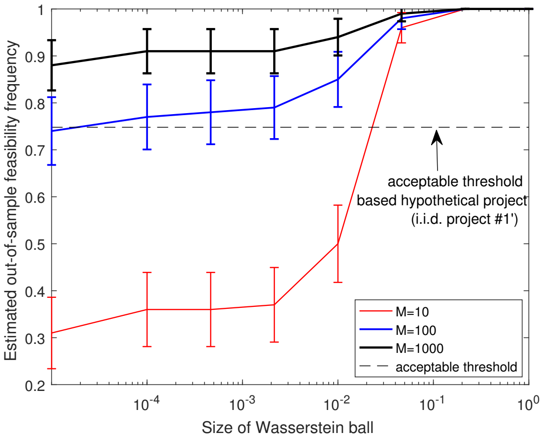
1000 samples

Synthetic Data: Out-of-Sample Feasibility of SOSD



- The estimated out-of-sample feasibility frequency shows the probability of obtaining a WDRSOSDCP solution that satisfies the SOSD constraint in out-of-sample test.

Synthetic Data: Out-of-Sample Feasibility of SOSD



- The estimated out-of-sample feasibility frequency shows the probability of obtaining a WDRSOSDCP solution that satisfies the SOSD constraint in out-of-sample test.

Real Stock Data: Computational Performance

Table: The average computational performance with respect to different ϵ, number of stocks ($m \in \{10, 50, 100\}$) and in-sample sizes ($M \in \{50, 100\}$), in terms of average CPU time (Time, in seconds), proportion of unsolved instances (prop, in $\%$), and average number of iterations (Iter) over 20 runs.

m		10			50			100		
M	ϵ	Time	prop	Iter	Time	prop	Iter	Time	prop	Iter
50	0.0100	242	0	6.0	1997	0.05[1.9]	5.0	3979	0.25[2.2]	5.0
	0.0464	103	0	5.0	3418	0.10[1.4]	6.0	5966	0.25[3.8]	6.0
100	0.0100	1528	0	6.0	6259	0.05[3.0]	6.0	6269	0.45[4.6]	5.0
	0.0464	506	0	5.0	5966	0.25[3.8]	6.0	5073	0.80[2.2]	6.0
Average		595	0	5.5	4410	0.11[2.5]	5.8	5322	0.44[3.2]	5.3

$[\cdot]$: the average sub-optimality gap (in $\%$) for the unsolved instances within 2 hours limit.
Based on our test fom the previous experiment, the midrange values of ϵ (i.e., $\epsilon \in \{0.0100, 0.0464\}$) appeared to be the hardest to handle.